Amenable Groups

Jake Bahr

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Amenable groups

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- reminder: $\mu(1) = 1$ and $\mu$ is positive

- Assume every group is discrete from here on out.
Two remarks

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2. We could have equivalently defined $G$ as amenable if there is a left-invariant finitely additive probability measure.
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1. Note that we could have chosen right-invariant and we would be considering the same groups.

2. We could have equivalently defined $G$ as amenable if there is a left-invariant finitely additive probability measure.
   - given a finitely additive measure $m$, the integral $\int \cdot dm$ is our invariant mean
   - given an invariant mean $\mu$, $m(A) = \mu(\chi_A)$ is our invariant finitely additive probability measure
Quick examples

Consider any finite group $G$. Then $\frac{1}{|G|} \sum_{g \in G} \delta_g$ is an invariant mean.
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- Consider any finite group $G$. Then $\frac{1}{|G|} \sum_{g \in G} \delta_g$ is an invariant mean.

- Extending our definition to locally compact groups, compact groups are amenable. The Haar measure is our left invariant mean (in the sense of measure).

- $\mathbb{Z}^n$ is amenable, and in fact every abelian group is amenable.
Følner Sequences

We say a discrete countable group $G$ has a Følner sequence if it has:

1. $F_n \nearrow G$
2. for every $g \in G$, we have $|g F_n \triangle F_n| \to 0$ as $n \to \infty$.

where $\triangle$ is the symmetric difference.

Informally, large $F_n$'s don't move much when pushed by any fixed element of $G$.

Equivalent to amenability.
Følner Sequences

We say a discrete countable group $G$ has a Følner sequence if it has: a sequence $\{F_n\}_{n=1}^{\infty}$ of finite subsets of $G$ satisfying:

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Informally, large $F_n$’s don’t move much when pushed by any fixed element of $G$.

Equivalent to amenability
An informal example

\[ \mathbb{Z}^n \] has a Følner sequence given by \( F_m = \{(z_1, \ldots, z_m) \mid |z_i| \leq m\} \).

- after perturbing this set by any element \( g \in \mathbb{Z}^n \), we see that only \( F_m \)'s "boundary" gets counted by \( |gF_m \triangle F_m| \), and the surface area of a box is small relative to the volume for large boxes.
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or:

\( \triangleright \) push a square just a bit: the leftovers are linear but the area is quadratic so the ratio goes to zero.
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Følner’s Theorem

A discrete *countable* group \( G \) has a Følner sequence iff it is amenable.
A first attempt at the forward direction

Let's try to show the existence of a left-invariant probability measure $\mu$. 
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\mu(A) := \lim_{n \to \infty} \frac{|F_n \cap A|}{|F_n|}.
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Example: consider $F_n = [-n, n]$ a Følner sequence for $\mathbb{Z}$. This definition yields the asymptotic density of $A$ in $\mathbb{Z}$. 
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▶ Example: consider $F_n = [-n, n]$ a Følner sequence for $\mathbb{Z}$. This definition yields the asymptotic density of $A$ in $\mathbb{Z}$.

▶ Problem: does the limit exist (and define something reasonable?)
If $G$ has a Følner sequence, it is amenable

Let $F_n$ be a Følner sequence for $G$ and define a finitely additive probability measure on $G$ by

$$\mu(A) = \lim_{\omega} \frac{|F_n \cap A|}{|F_n|}$$

where $\omega$ is a nonprincipal ultrafilter on $\mathbb{N}$. 
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$\mu(G) = 1$ because the limit exists and ultralimits agree with limits.
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  \mu(A) = \lim_\omega \frac{|F_n \cap A|}{|F_n|}
  $$
  where $\omega$ is a nonprincipal ultrafilter on $\mathbb{N}$.
- $\mu(G) = 1$ because the limit exists and ultralimits agree with limits.
- We have left invariance
  $$
  |\mu(gA) - \mu(A)| \leq \frac{1}{|F_n|}||F_n \cap gA| - |F_n \cap A||
  \leq \frac{1}{|F_n|}|(g^{-1}F_n \Delta F_n) \cap A| \to 0
  $$
Følner’s Theorem

Let’s show now that countable discrete amenable groups have Følner sequences. We’ll do this in a few steps using an argument of Namioka:
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2. Given a finite set $S$ and $\varepsilon > 0$, we can find a large $F$ such that

$$\frac{|gF \triangle F|}{|F|} \leq \varepsilon$$

for each $g \in S$
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2. Given a finite set $S$ and $\varepsilon > 0$, we can find a large $F$ such that

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for each $g \in S$
3. The existence of a Følner sequence follows.
Følner’s Theorem: Step 1

Suppose $G$ is an amenable countable discrete group. Let $\mu \in \ell^\infty(G)$ be a left invariant mean. We’d like to show that for all finite sets $S \subseteq G$ and $\varepsilon > 0$, there exists a finite mean, i.e., a finitely supported function $\nu : G \rightarrow \mathbb{R}^+$ with $\|\nu\|_{\ell^1(G)} = 1$ satisfying:
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$$\|\nu - L_g \nu\|_{\ell^1(G)} < \varepsilon$$
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Suppose not. Then there exists $S \subseteq G$ finite and $\varepsilon > 0$ with $\sup_{g \in S} \|\nu - L_g \nu\|_{\ell^1(G)} \geq \varepsilon$ for every finite mean $\nu$. 

Amenable Groups
Følner’s Theorem: Step 1

Set $V = \{ \nu \in \ell^1(G) \mid \nu \text{ is a finite mean} \}$. 
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- Set $V = \{ \nu \in \ell^1(G) \mid \nu \text{ is a finite mean}\}$.
- Taking the norm $\sup_{g \in S} \| f(g, \cdot) \|$ for the space $(\ell^1(G))^S$,
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- Set $V = \{ \nu \in \ell^1(G) \mid \nu \text{ is a finite mean} \}$.
- Taking the norm $\sup_{g \in S} \| f(g, \cdot) \|$ for the space $(\ell^1(G))^S$, note that $\{(\nu - L_g \nu)_{g \in S} \mid \nu \in V\}$ is a convex subset of $V^S \subseteq (\ell^1(G))^S$ bounded away from zero.
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- Set $V = \{ \nu \in \ell^1(G) \mid \nu \text{ is a finite mean} \}$.
- Taking the norm $\sup_{g \in S} \| f(g, \cdot) \|$ for the space $(\ell^1(G))^S$, note that $\{(\nu - L_g \nu)_{g \in S} \mid \nu \in V\}$ is a convex subset of $V^S \subseteq (\ell^1(G))^S$ bounded away from zero.
- Hahn-Banach separation yields a functional $\alpha \in ((\ell^1(G))^S)^*$, with

$$\alpha_g(\nu - L_g \nu) > 1$$

for all $\nu \in V$ and $g \in S$ where we write $\alpha$ as $(\alpha_g)_{g \in S}$. 
Følner’s Theorem: Step 1

Rewriting our previous condition, using the fact that 
\(((\ell^1(G))^S)^* \simeq (\ell^\infty(G))^S\), we get for each \(g \in S\) a function 
\(\beta_g \in \ell^\infty(G)\) satisfying

\[
\sum_x \beta_g(x)(\nu(x) - (L_g\nu)(x)) > 1
\]

for all finite means \(\nu\).
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  $$\sum_x \beta_g(x) (\nu(x) - (L_g\nu)(x)) > 1$$

  for all finite means $\nu$.

- Let’s consider $\nu = \delta_h$ to get

  $$\beta_g(h) - (L_g^{-1}\beta_g)(h) > 1$$

  which holds for every $h \in G$. 

Følner sequence implies amenability

Amenable groups have Følner sequences
Følner’s Theorem: Step 1

For each $g \in S$,

$$\beta_g(h) - (L_g^{-1}\beta_g)(h) > 1$$

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Følner sequence implies amenability
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Følner’s Theorem: Step 1

For each \( g \in S \),

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\beta_g(h) - (L_{g^{-1}} \beta_g)(h) > 1
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for all \( h \in G \), so

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\mu(\beta_g - L_{g^{-1}} \beta_g) > \mu(1) = 1,
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a contradiction to left-invariance.
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a contradiction to left-invariance.

Thus given our invariant mean \( \mu \), it’s true that for all \( S \) finite and \( \varepsilon > 0 \), there’s a finite mean \( \nu \) satisfying \( \forall g \in S \),

\[
\|\nu - L_g\nu\|_{\ell^1(G)} < \varepsilon
\]
Følner’s Theorem: Step 2

Goal: for every $S$ finite and $\varepsilon > 0$ there exists $F$ finite satisfying

$$\frac{|gF \triangle F|}{|F|} \leq \varepsilon \quad \forall g \in S.$$
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Let $S$ be finite and $\varepsilon > 0$. By Step 1, we have a finite mean $\nu$ with

$$\|\nu - L_g\nu\|_{\ell^1(G)} \leq \frac{\varepsilon}{|S|}$$

for all $g \in S$. 

"Følner sequence implies amenability

Amenable groups have Følner sequences"
Følner’s Theorem: Step 2

Since \( \nu \) is finitely supported, let’s take its "layer cake decomposition":

\[
\nu = \sum_{i=1}^{n} c_i \chi_{F_i}
\]

with \( c_i > 0 \) and \( F_1 \supseteq \cdots \supseteq F_n \).
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$$\nu = \sum_{i=1}^{n} c_i \chi_{F_i}$$

with $c_i > 0$ and $F_1 \supseteq \cdots \supseteq F_n$. Note that $\sum c_i |F_i| = 1$ since $\nu$ is a finite mean.
Følner’s Theorem: Step 2

Note that $|\nu(g) - (L_h\nu)(g)| \geq c_i$ for $g \in hF_i \triangle F_i$. Think of hopping up or down a layer of the cake.
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Note that $|\nu(g) - (L_h\nu)(g)| \geq c_i$ for $g \in hF_i \triangle F_i$. Think of hopping up or down a layer of the cake. Integrating the above,

$$\sum_{i=1}^{n} c_i |hF_i \triangle F_i| \leq \|\nu - L_h\nu\|_{\ell^1(G)} \leq \frac{\varepsilon}{|S|} \sum_{i=1}^{n} c_i |F_i|$$

using $\sum c_i |F_i| = 1$.  

Følner sequence implies amenability
Amenable groups have Følner sequences
Følner’s Theorem: Step 2

Summing in $h \in S$,

$$\sum_{i=1}^{n} \sum_{h \in S} c_i |hF_i \triangle F_i| \leq \varepsilon \sum_{i=1}^{n} c_i |F_i|$$
Følner’s Theorem: Step 2

Summing in \( h \in S \),

\[
\sum_{i=1}^{n} \sum_{h \in S} c_i |hF_i \triangle F_i| \leq \varepsilon \sum_{i=1}^{n} c_i |F_i|
\]

By pigeonhole, there must exist \( i \) with

\[
\sum_{h \in S} \frac{|hF_i \triangle F_i|}{|F_i|} \leq \varepsilon
\]

This \( F_i \) is our desired set.
Følner’s Theorem: Step 3

Now we know that for all $S$ and $\varepsilon > 0$ there exists $F$ satisfying $|gF \triangle F|/|F| \leq \varepsilon$ for all $g \in S$, we’ll find a Følner sequence. For each $\varepsilon = 1/n$ select $F_n$ satisfying the above.
Quotients of amenable groups are amenable

Let $H = G/N$ be a quotient of an amenable discrete group. Given $\mu$ on $G$, define $\nu$ on $H$ by

$$\nu(A) = \mu(AN)$$
If a normal subgroup and the quotient are amenable, so is the original

Suppose $N \trianglelefteq G$ is amenable and $G/N$ is amenable. Let $\mu$ and $\nu$ be measures on $N$ and $G/N$ respectively.
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Suppose $N \trianglelefteq G$ is amenable and $G/N$ is amenable. Let $\mu$ and $\nu$ be measures on $N$ and $G/N$ respectively.

Define $f_A : G \to \mathbb{R}$ via $f_A(g) = \mu(N \cap g^{-1}A)$ for $A \subseteq G$. 

▶ Pull back to $f_A : G/N \to \mathbb{R}$, noting $\mu$ is $N$-invariant.

▶ Define $\psi(A) = \int f_A(x) d\nu(x)$. 

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Amenable Groups
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Subgroups of amenable groups are amenable

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Subgroups of amenable groups are amenable

Let $H$ be a subgroup of $G$ (discrete). By the axiom of choice, let $S$ contain precisely one element of every right coset of $H$. Given a probability measure $\mu$ on $G$, define $\nu$ on $H$ via $\nu(A) = \mu(AS)$. 
Discrete abelian groups are amenable

This fact follows from a few things:

- Direct limits of amenable groups are amenable
- Direct sums of amenable groups are amenable
Free groups (except $\mathbb{Z}$) are not amenable

- Picture the Cayley graph of $F_2$. Any large set has very large "boundary", so informally we should be concerned about the existence of a Følner sequence.

- Since subgroups of amenable groups are amenable, it suffices to prove that $F_2$ is not amenable.
A Banach-Tarski Trick

Let $F_2 = \langle a, b \rangle$ and denote by $W(x)$ the words beginning with $x$. 
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Let $\mathbb{F}_2 = \langle a, b \rangle$ and denote by $W(x)$ the words beginning with $x$. We can decompose

$$\mathbb{F}_2 = W(a) \sqcup aW(a^{-1})$$

$$= W(b) \sqcup bW(b^{-1})$$
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If $F_2$ had a left-invariant probability measure:
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$$F_2 = W(a) \sqcup aW(a^{-1}) = W(b) \sqcup bW(b^{-1})$$

If $F_2$ had a left-invariant probability measure:

1. $\mu(W(a)) + \mu(W(a^{-1})) = 1$, and so $\mu(W(b)) = \mu(W(b^{-1})) = 0$. 
A Banach-Tarski Trick

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If $\mathbb{F}_2$ had a left-invariant probability measure:

1. $\mu(W(a)) + \mu(W(a^{-1})) = 1$, and so $\mu(W(b)) = \mu(W(b^{-1})) = 0$.

2. Similarly for $b$. 
Sources

Mostly

- [http://reh.math.uni-duesseldorf.de/~garrido/amenable.pdf](http://reh.math.uni-duesseldorf.de/~garrido/amenable.pdf)
- [https://terrytao.wordpress.com/2009/04/14/some-notes-on-amenability/](https://terrytao.wordpress.com/2009/04/14/some-notes-on-amenability/)