Analysis Qual Topics

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Contents

1 Real Analysis

1.1 Basic Analysis Facts

1.1.1 Semi-continuity

1. Lower semi-continuous functions

   A function is lower semicontinuous if and only if $\lim_{y\to x} f(y) \leq f(x)$ for all x. It turns out that if f is bounded from below and is lower semi-continuous, it is the pointwise increasing limit of continuous functions.

   Define $f_n(x) = \inf\{f(y) + kd(x,y)\}$.

1.1.2 Convolutions on $\mathbb{R}^n$:

The convolution is $f * g(x) = \int f(x-y)g(y)\,dy$. It's symmetric in f and g by a change of variables. The following properties hold (assuming integrals exist) by Folland p.240:

1. commutativity
2. associativity
3. $\tau_z(f*g) = (\tau_z f)*g = f*(\tau_z g)$
4. $\mathrm{supp}(f*g) \subseteq \overline{\mathrm{supp}(f)+\mathrm{supp}(g)}$

where τ_z is the translation operator.

Also of note is the conintuity of a convolution. Let $f \in L^\infty$ and $K \in L^1$. Then $f * K$ is uniformly continuous (and bounded by Young's inequality).

If K is C_c, then

$$|(f*K)(x) - (f*K)(y)| \leq \int |f(z)||K(x-z) - K(y-z)|\,dz$$

is enough by uniform continuity of K.

Otherwise we use a density argument. If $K_n \to K$ in L^1, then $(f*K_n) \to f*K$ uniformly. Since the uniform limit of uniformly continuous functions is uniformly continuous, we're done.

1.1.3 Translation Operator

The translation operator is $\tau_y f(x) = f(x-y)$. For $1 \le p < \infty$, translation is continuous in L^p, i.e., $\|\tau_y f - f\|_{L^p} \to 0$ as $y \to 0$. This obviously fails in L^∞.

Proof: if $g \in C_c$, then for $|y| \le 1$, τ_g are supported in a common compact set, so uniformly continuous. Now approximate by C_c functions and note that the L^p norm is translation invariant. The result holds for $p \neq \infty$ because C_c is not dense in L^∞.

1.1.4 Approximate Identities and Mollification

With η_ε and f_ε as usual, we note that for $f \in L^1_{\text{loc}}$, $f_\varepsilon \to f$ almost everywhere (specifically at every Lebesgue point), and in fact converges locally uniformly if f is continuous. Furthermore, the convergence is in L^p_{loc} for every $1 \le p < \infty$ if $f \in L^p_{\text{loc}}$.

Proof: for ae convergence, write $f_\varepsilon(x) - f(x) = \int \eta_\varepsilon(x-y)(f(y)-f(x))$. Apply Lebesgue differentiation. If f is continuous, then it's uniformly continuous on a smaller (compact) set, so the limit is uniform (since Lebesgue differentiation is uniform).

For L^p: $|f_\varepsilon(x)| \le \int \eta^{1/p'}\eta^{1/p}|f|$, so apply H\"older and get $|f_\varepsilon(x)| \le \left\|\eta_\varepsilon |f|^p\right\|_{L^p(B(x,\varepsilon))}$. Then $\|f_\varepsilon\| \le \|f\|$ in L^p so an approximation argument works to prove convergence of $f_\varepsilon \to f$ in L^p.

1.1.5 Stone-Weierstrass

1. Polynomials are (sup-norm) dense in $C([0,1])$.

   Proof: convolve with $c_n(1-x^2)^n$ as a kernel. Some technical bounds are needed. It's not great.

2. Compact Hausdorff

   Generalization: Let X be compact Hausdorff. If $A \subseteq C(X;\mathbb{R})$ is a subalgebra that contains a constant function, then

   $$A \text{ separates points} \iff A \subseteq C(X) \text{ is dense}$$

3. Locally compact Hausdorff

   Let X be locally compact Hausdorff. If $A \subseteq C_0(X;\mathbb{R})$ is a subalgebra (where C_0 is the space of continuous functions that vanish at infinity) then

   $$A \supseteq C_0(X) \text{ is dense} \iff A \text{ separates points and vanishes nowhere}$$

where A vanishes nowhere if $\forall x \in X.\ \exists f \in A.\ f(x) \neq 0$.

Note that this easily implies the compact version (with "vanishes nowhere" instead of "contains the constants"). Furthermore, to easily prove this version from the compact version (with "vanishes nowhere"), prove the following lemma:

Lemma: if X is compact Hausdorff and A separates points but vanishes at x_0, then the closure is $\{f \in C(X;\mathbb{R}) \mid f(x_0) = 0\}$.

Proof: one direction is trivial. For the other, throw in $+c1$ where $c \in \mathbb{R}$ and apply the compact version.

Finally, apply this to $C(X^*;\mathbb{R})$ where X^* is the one point compactification.

1.1.6 Baire Category Theorem

The intersection of countably many open dense sets is dense.

Proof: telescope/induct (find points in U_1 then $U_1 \cap U_2$, etc.).

Corollary: Banach spaces aren't meager.

1.1.7 Arzela-Ascoli (normal families)

Equicontinuity and uniform pointwise boundedness on a compact metric space implies precompactness as a subset of $C(X)$.

Proof: diagonalization.

1.1.8 Useful Inequalities

1. Hölder's inequality

   $\|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^{p'}}$ where $1/p + 1/p' = 1$.

   Generalized version: $\|\prod f_i\|_{L^1} \leq \prod \|f_i\|_{L^{p_i}}$ where $\sum 1/p_i = 1$.

2. Integrals of L^1 functions on small sets

   Integrals of L^1 functions on small sets are small. Using Hölder doesn't work in this case, because $\int |f| 1 \leq \|f\|_{L^1}\|1\|_{L^\infty}$ is useless.

   Formally, it's a kind of continuity: $\forall \varepsilon > 0.\ \exists \delta > 0.\ m(E) < \delta \implies \int_E |f| < \varepsilon$.

   Proof: consider a simple function with integral at most $\varepsilon/2$ away. The claim is trivial for simple functions.

3. Minkowski's Inequalities

   The usual Minkowski inequality is just the triangle inequality on L^p.

   Minkowksi's inequality for integrals is as follows:

$$\left(\int_Y \left|\int_X f(x,y)\,d\mu(x)\right|^p d\mu(y)\right)^{1/p} \leq \int_X \left(\int_Y |f(x,y)|^p\,d\mu(y)\right)^{1/p} d\mu(x),$$

   i.e., $\left\|\int f\,dx\right\|_{L^p(dy)} \leq \int \|f\|_{L^p(dy)}\,dx$.

4. Chebyshev's Inequality

   The inequality: $|\{f > \lambda\}| \leq \|f\|_{L^p}^p/\lambda^p$.

   Proof: consider a characteristic function $\lambda\chi_{f>\lambda} \leq f$ and take L^p norms.

5. Jensen's Inequality If $g : X \to (a,b)$ is integrable, ϕ is convex, and $\mu(X) = 1$, then

$$\phi\left(\int g\right) \leq \int \phi \circ g$$

   Proof: Note that ϕ is AC on every compact subinterval of (a,b) and its derivative is increasing (where it's defined). Prove that $\phi(t) - \phi(t_0) \geq \beta \cdot (t - t_o)$ for all $t \in (a,b)$ and some $\beta \in \mathbb{R}$.

   Finally, let $t_0 = \int g\,d\mu$ and $t = g(x)$ in the above expression and integrate. See Folland p109.

   Note: do not confuse with Jensen's formula for the value of $\log|f(0)|$ *for a harmonic function* f.

6. Young's Convolution Inequality

   For $f \in L^p$ and $g \in L^q$, and

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$$

   with $1 \leq p, q, r \leq \infty$. Then we have $\|f * g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}$.

   There are some special cases with $L^{p,w}$:

   (a) Suppose further that $p, q > 1$, and $r < \infty$. If $f \in L^p$ and $g \in L^{q,w}$, then $f * g \in L^r$ and

$$\|f * g\|_{L^r} \leq C_{p,q}\|f\|_{L^p}\|g\|_{L^{q,w}}^*$$

   where $C_{p,q}$ is independent of f and g.

(b) Suppose instead that $p = 1$ and $r = q > 1$. If $f \in L^1$ and $g \in L^{q,w}$, then $f * g \in L^{q,w}$, and

$$\|f * g\|^*_{L^{q,w}} \le C_q \|f\|_{L^1}$$

1.1.9 Tricks

1. Differentiation under the integral sign

   If $X \subseteq \mathbb{R}$ and Y is a measure space, and $f : X \times Y \to \mathbb{R}$ satisfies

   (a) $f(x, y)$ is integrable in y for each x

   (b) For almost every y, $\frac{\partial f}{\partial x}$ exists

   (c) There's an integrable function $m : Y \to \mathbb{R}$ such that $\left|\frac{\partial f}{\partial x}\right| \le m(y)$ for almost every y and all x.

   then by Dominated Convergence, $\frac{d}{dx} \int f(x, y) = \int \frac{\partial f}{\partial x}$.

2. Showing finite almost everywhere

   Think about showing L^1_{loc} or L^p_{loc} instead.

1.2 Basic Topology

1.2.1 Urysohn's Lemma

In a normal space, two closed disjoint sets can be separated by a function (1 on one, 0 on the other). Proof: Define an "onion" function, ((dyadic?) rationally-indexed sets in between the two sets and then interpolate somehow) Proof isn't important.

1.3 Fourier Analysis

1.3.1 Fourier Transform Notation

We use the notation $\hat{f}(\xi) = \int f(x) e^{-2\pi i \xi \cdot x}\, dx$.

For series, we have $(n) = 1 $\frac{}{2\pi \int_0^{2\pi}}$ f(x) e^{-inx} \,dx

1.3.2 Fourier Inversion

If $f, \hat{f}$ are in L^1, then $f = f_0$ ae for some continuous function f_0 and $F^{-1}(F(f)) = f_0 = F(F^{-1}(f))$. This works for series too.

We write $f(x) = \int \hat{f}(\xi) e^{2\pi i \xi \cdot x}\, d\xi$.

1.3.3 Fourier Series and Convergence of Trigonometric Polynomials

A trigonometric polynomial (of degree N) on $[0, 2\pi]$ is a finite series of the form $p(x) = \sum_{n=-N}^{N} c_n e^{inx}$. Given $f \in L^1(\mathbb{T})$, we can write $f_N = \sum_{n=-N}^{N} \hat{f}(n) e^{inx}$ where $\hat{f}(n) = ⨍_0^{2\pi} f(x) e^{-inx}\, dx$.

If we're working on $\mathbb{T} = S^1$ in the complex plane, a trigonometric polynomial becomes $\sum_{n=-N}^{N} c_n z^n$.

Note that trigonometric polynomials are dense (uniformly) in continuous functions by Stone Weierstrass. This is often as much as you need from this section of material.

1. If $f \in L^2(\mathbb{T})$, then $f_N \to f$ in L^2
2. If $f \in C^1(\mathbb{T})$, then $f_N \to f$ in L^∞ (and in fact pointwise).
   (a) This is easier to show if $f \in C^2$ by noting that $n^2 \hat{f}(n) \to 0$ as $n \to \infty$.

Normally, the Fourier series is considered on $\mathbb{T}^n$

1.3.4 Riemann-Lebesgue Lemma

$\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$ and when $f \in L^1$, we have $\hat{f} \in C_0$.

Proof: $\left|\hat{f}(x)\right| \leq \|f\|_{L^1}$ is trivial. Let $\mathcal{S} \ni f_n \to f$. Then $\hat{f}$ is the uniform limit of Schwartz functions, and hence C_0.

1.3.5 Plancherel and Parseval

Plancherel: $\mathcal{F}$ preserves norm on L^2. This follows immediately from Parseval's identity, that $\mathcal{F}$ is unitary.

This itself follows from the fact that $\langle f, \hat{g}\rangle = \left\langle \hat{f}, g\right\rangle$ (from Fubini) and Fourier inversion. These results hold for $\mathcal{S}$ and thus by density, hold in L^2.

A different way to phrase Parseval's identity is that for $f, g \in H$ some separable Hilbert space with orthonormal basis e_n, we have

$$\langle f, g\rangle = \sum \langle f, e_n\rangle \overline{\langle g, e_n\rangle}.$$

1.3.6 Hausdorff-Young

Let $1 \leq p \leq 2$ and $1/p + 1/q = 1$. If $f \in L^p$, then $\hat{f} \in L^1$ and

$$\|\hat{f}\|_{L^q} \leq \|f\|_{L^p}$$

Follows because $\mathcal{F}$ is strong $(1, \infty)$ by Riemann Lebesgue and strong $(2, 2)$ by Plancherel, so Riesz–Thorin gives us strong (p, q). Since the constants depend on the endpoint constants (both 1), we get this inequality exactly. Marcinkiewicz (on Lorentz spaces) gives us the result for $L^{p,\infty}$ functions.

1.3.7 Poisson Summation

If $f \in C(\mathbb{R}^n)$ satisfies $|f| \lesssim \langle |x| \rangle^{-n-\varepsilon}$ and $\left|\hat{f}\right| \lesssim \langle |\xi| \rangle^{-n-\varepsilon}$ for some $\varepsilon > 0$, then

$$\sum_{k \in \mathbb{Z}^n} f(x + k) = \sum_{\kappa \in \mathbb{Z}^n} \hat{f}(\kappa) e^{2\pi i \kappa \cdot x}$$

where both series converge absolutely and uniformly on $\mathbb{T}^n$.

In particular, taking $x = 0$, we get

$$\sum_{k \in \mathbb{Z}^n} f(k) = \sum_{\kappa \in \mathbb{Z}^n} \hat{f}(\kappa)$$

Proof: Absolute and uniform convergence follows by the bounds on the growth by the integral test.

The function $Pf = \sum_k \tau_k f \in C(\mathbb{T}^n)$ and hence in $L^2(\mathbb{T}^n)$. The following theorem shows that $\sum \hat{f}(\kappa) e^{2\pi i \kappa \cdot x}$ converges to Pf in L^2 (by writing it as a Fourier series). Since the convergence is uniform also, its sum is Pf pointwise.

1. Requisite Theorem

   If $f \in L^1(\mathbb{R}^n)$, the series $\sum_k \tau_k f$ converges pointwise ae and in L^1 to a function Pf such that $\|Pf\|_{L^1} \leq \|f\|_{L^1}$.

   Moreover, for $\kappa \in \mathbb{Z}^n$, $\widehat{(Pf)}(\kappa)$, the Fourier transform on $\mathbb{T}^n$, equals $\hat{f}(\kappa)$, the Fourier transform on $\mathbb{R}^n$.

1.3.8 Properites of the Fourier transform

1. $f, g \in L^2$ implies $(\hat{f}\hat{g})^\vee = f * g$
2. If $f, g \in L^1$, then (Folland p.249):
   (a) $\widehat{\tau_y f}(\xi) = e^{-2\pi i \xi \cdot y} \hat{f}(\xi)$
   (b) $\tau_y(\hat{f}) = \hat{h}$ where $h = e^{2\pi i y \cdot x} f(x)$

(c) If $T \in \mathrm{GL}(n)$, and $S = (T^*)^{-1}$ is the inverse transpose, then

$$\widehat{f \circ T} = \left|\det T\right|^{-1} \hat{f} \circ S$$

(and if T is a rotation then $T = S$.

(d) $\widehat{f * g} = \hat{f}\hat{g}$.

(e) If $x^\alpha f \in L^1$ for all $|\alpha| \le k$, then $\hat{f} \in C^k$ and

$$\partial^\alpha \hat{f} = ((\widehat{-2\pi i x)^\alpha} f)$$

In other words, $(-2\pi i x)^\alpha$ is the multiplier corresponding to ∂^α.

(f) If $f \in C^k$ and $\partial^\alpha f \in \mathscr{L}^1$ for $|\alpha| \le k$ and $\partial^\alpha f \in C_0$ for $|\alpha| \le k - 1$, then

$$\widehat{\partial^\alpha f}(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi)$$

1.3.9 Some Interesting Kernels

1. Gaussian kernel (also called the Gauss kernel, the Weierstrass kernel, or just a bell curve or normal curve or distribution): $e^{-\pi|\xi|^2}$ is its own Fourier transform (no scaling factors!).

$$\int_{\mathbb{R}^n} e^{-\pi|\xi|^2} e^{-2\pi i \xi \cdot x} \, d\xi = \prod_{i=1}^{n} \int_{\mathbb{R}} e^{-\pi \xi_i^2} e^{-2\pi i \xi_i x_i} \, dx_i$$

Now we just need the Fourier transform in $\mathbb{R}$. Let $f(\xi) = \int e^{-\pi\xi^2} e^{-2\pi i \xi x} \, dx$ for $x, \xi \in \mathbb{R}$. Differentiate both sides in terms of ξ, view this as a derivative in x on the right side, and integrate by parts. We get an ODE with a unique solution.

More generally, $\mathscr{F}(e^{-a\pi|x|^2}) = \sqrt{\frac{1}{a}} e^{-\pi|\xi|^2/a}$.

2. The Poisson kernel: Let $\Phi(\xi) = e^{-2\pi|\xi|}$. Its inverse Fourier transform is the Poisson kernel in $\mathbb{R}^n$.

In $\mathbb{R}^1$, we get $\frac{1}{\pi(1+x^2)}$. This should remind us of the Poisson kernel used to find harmonic functions on the upper half-plane given boundary values:

$$u(x + iy) = \int_{-\infty}^{\infty} P_y(x - t) f(t) \, dt$$

where $P_y(x) = \frac{y}{\pi(y^2+x^2)}$.

In $\mathbb{R}^n$ the inverse fourier transform is a constant multiple of $\frac{1}{(1+|x|^2)^{\frac{n+1}{2}}}$.

This matches with the Poisson kernel in the upper half-space.

1.4 Functional Analysis

1.4.1 Hahn Banach

Let ϕ be a linear map on a subspace which is dominated by a seminorm p. Then there exists a linear extension to the whole space dominated by the same seminorm.

Proof: For sublinear functionals, first show that you can extend in a single direction. Let x be a new vector and consider $\sup\{f(y) - p(y - x)\} \leq \alpha \leq \inf\{p(x+y) - f(y)\}$ for y in the subspace. Then set $g(y + \lambda x) = f(y) + \lambda\alpha$. Then Zorn is up. For the complex version, note that a complex functional can be obtained from its real part by $F(x) = \operatorname{Re}(F(x)) - i \operatorname{Re}(F(ix))$.

Separation/geometric version: Let $K = \mathbb{R}$ or $\mathbb{C}$ and let V be a topological vector space over K. If A, B are convex and disjoint with A open, then there is a continuous linear map (whose real part) separates A and B.

Proof: use the Minkowski functional, $p_E(x) = \inf\{t \mid x/t \in E\}$.

1.4.2 Open Mapping Theorem

A surjective map between Banach spaces is an open map.

Proof: the image is the union of $T(B(0, n))$. Baire Category Theorem says that one of these is not nowhere dense. This gives us a ball inside $\overline{T(B_1)}$ after some convexity stuff.

If $\|y\| < r/2$ we can find $x_1 \in B(0, 1/2)$ that's closer to y than $r/4$. This difference is small and can be approximated by $x_2 \in B(0, 1/4)$. Inducting in this nested way and applying completenss gives us that y is in the image.

1.4.3 Closed Graph Theorem

If X, Y are Banach, then a linear map $T : X \to Y$ is closed (i.e., has a closed graph) iff T is bounded.

Proof: Bounded implies closed is trivial. For the other direction, the projection $\pi_1 : \Gamma(T) \to X$ is a linear bijection between Banach spaces, hence the inverse is bounded by open mapping. Then $T = \pi_2 \circ \pi_1^{-1}$.

1.4.4 Uniform Boundedness Principle (Banach-Steinhaus)

Let X and Y be normed vector spaces with $A \subseteq L(X, Y)$.

If $\sup_A \|Tx\| < \infty$ for all x in some non-meager subset of X, then $\sup_A \|T\| < \infty$.

If X is Banach and $\sup_A \|Tx\| < \infty$ for all x, then $\sup_A \|T\| < \infty$. The first version implies the second by Baire Category.

Proof: $E_n = \{x \mid \sup_A \|Tx\| \leq n\} = \bigcap_{T \in A} \{x \mid \|Tx\| \leq n\}$. These sets are closed, and so eventually one contains a ball by Baire Category. The result follows immediately.

1.4.5 Various Topologies

1. The strong topology is the norm topology.

2. The weak topology is the weakest (smallest) topology that makes all (norm) continuous linear functionals continuous. Convergence here is $x_\alpha \to x$ as nets iff $f(x_\alpha) \to f(x)$ as nets for all $f \in X^*$.

3. The weak star topology is $\sigma(X^*, X)$, i.e., convergence of $f_\alpha \to f$ as nets iff $f_\alpha(x) \to f(x)$ for all $x \in X$. In other words, it's the weakest topology that makes the evaluations maps continuous.

1.4.6 Separability

If X' is separable, then X is separable.

Proof: take a countable dense subset and take vectors with $|f_n(x_n)| \geq \frac{1}{2}\|f_n\|$ and apply Hahn-Banach to show that the $\{x_n\}$ are a countable dense subset of X.

Consider: $L^\infty = (L^1)^*$ and L^∞ is usually not separable but L^1 is.

Another fact: a totally bounded metric space is separable, so a non-separable space is not totally bounded. Proof is trivial, but this comes in handy.

1.4.7 Banach Alaoglu

The closed unit ball in X^* is weakly star compact.

Proof: Let $D_x = \{z \in \mathbb{C} \mid |z| \leq \|x\|\}$. Then $D = \prod_{x \in X} D_x$ can be identified with the unit ball in X^* by $f \mapsto (f(x))_{x \in X}$. The former is compact (Tychonoff), so the latter is too.

Note: if X is separable, then X^* is metrizable, and so weakly-star compactness and weakly-star sequential compactness are identical.

1. Tychonoff's Theorem

   Every product of compact topological spaces is compact. The proof is messy and involves the axiom of choice.

1.4.8 Adjoint of a Linear Operator

If $T : V \to W$ is linear (resp.\ and continuous), then $T^* : W^* \to V^*$ is too.

1. The kernel of T^* is the annihilator of the range of T. Thus T^* is injective if and only if the annihilator of the range is empty, i.e., the range is dense.

2. The kernel of T is the annihilator of the image of T^*. Thus T is injective if and only if the range of T^* is dense.

1.4.9 Spectrum of a self-adjoint linear operator on a Hilbert space

The spectrum is real. I would be money this doesn't appear on the exam.

1.4.10 L^p spaces

All are reflexive (i.e., isomorphic to their double dual via the canonical injection $x \mapsto \langle \cdot, x \rangle$) except the endpoints. We have $(L^1)^* = L^\infty$, but the dual of L^∞ is large and gross.

These spaces are all separable for $p < \infty$ (where the counter-example for L^∞ is obvious) when taken over a separable measure space.

1.4.11 Weak L^p spaces

The distribution function of f is $\lambda_f(t) = \mu\{|f| > t\}$. We define $\|f\|_{L^{p,w}} = \sup_t t(\lambda_f(t))^{1/p}$. This is not a norm (it's a quasinorm, where the triangle inequality loses a constant), but it is equivalent to a norm.

$L^{p,w} = L^{p,\infty}$, the Lorentz space.

1.4.12 Lorentz spaces

We define the quasi-norm $\|f\|^*_{L^{p,q}} = p^{1/q} \left\| t(\lambda_f(t))^{1/p} \right\|_{L^q(\mathbb{R}^+, dt/t)}$.

Note that $L^{p,p} = L^p$ and $L^{p,q} \subseteq L^{p,r}$ for $q < r$.

1.4.13 Distributions

$C_c^\infty(U)$ is given a metric by the semi-norms $\sup_{K\subseteq U}|\partial^\alpha f|$ where K is compact. This is a Frechet space, with dual $\mathscr{D}'$, the space of distributions. We can define for $T \in \mathscr{D}'$:

1. $\partial^\alpha T(f) = (-1)^{|\alpha|}T(\partial^\alpha f)$
2. $\tau_y T(f) = T(\tau_{-y} f)$
3. $gT(f) = T(gf)$ for g smooth
4. $T \circ S(f) = \left|\det S\right|^{-1} T(f \circ S^{-1})$ where S is linear
5. $(T * \psi)(f) = T(f * \tilde{\psi})$ where $\tilde{\psi}$ is the composition of ψ with the reflection.

1. Schwartz Functions

   A function s is Schwartz, i.e., $s \in \mathcal{S}$ iff $|\partial^\alpha \phi|\langle x\rangle^N$ is bounded for each N and α. Their suprema produce a family of semi-norms, ensuring $\mathcal{S}$ is Frechet.

   The dual of $\mathcal{S}$ is the space of tempered distributions.

2. Compactly Supported Distributions

   The dual of C^∞ is the space $\mathscr{E}'$ of compactly supported distributions. The Fourier transform of such a distribution is smooth (and analytic if the domain is $\mathbb{C}$).

   Proof: differentiate under the integral sign.

   Furthermore by the (Schwartz-)Paley-Weiner theorem, a compactly supported distribution is actually C_c^∞ if and only if its Fourier transform decays like $\langle\xi\rangle^{-N}$ as $\xi \to \infty$ for all N.

   Compactly supported distributions are automatically tempered.

   - Paley-Wiener Theorems
     See Strichartz's *A Guide To Distribution Theory* for more details, around p112. *Note: best just to know that Fourier transforms of compactly supported distributions are analytic.*
     If T is a compactly supported distribution, its Fourier transform is $O(\langle\xi\rangle^N \exp(|A\,\mathrm{Im}(\xi)|))$ for some N and A, and the Fourier transform is analytic on $\mathbb{C}$. (In fact, the compact support and A are related.)
     Conversely, analytic functions bounded as such are Fourier transforms of compactly supported distributions.

1.4.14 Interpolation

1. Riesz-Thorin (Complex Interpolation)

   If T ($\mathbb{R}$ or $\mathbb{C}$-linear!) is strong (p_0, q_0) and strong (p_1, q_1), then T is strong (p_θ, q_θ) and

   $$\|T\|_{L^{p_\theta} \to L^{q_\theta}} \le \|T\|^{1-\theta}_{L^{p_0} \to L^{q_0}} \|T\|^{\theta}_{L^{p_1} \to L^{q_1}}.$$

   The proof relies on the Hadamard three-lines lemma (hence the name "complex").

2. Marcinkiewicz (Real Interpolation) *This is not necessary, but helpful in the weak L^1 and (strong) L^∞ case.*

   In order of increasing generality:

   (a) Let T (merely $\mathbb{R}$ or $\mathbb{C}$-sublinear!) be weak-type (p, p) and weak-type (q, q). Then T is strong (r, r) for all $p < r < q$. We have the bound:

   $$\|Tf\|_{L^r} \lesssim_{p,q,r} \left(\|T\|^*_{L^{p,w}}\right)^{\theta} \left(\|T\|^*_{L^{q,w}}\right)^{1-\theta} \|f\|_{L^r}$$

   where $\|\cdot\|^*_{L^{p,w}}$ is the quasi-norm for the Lorentz space $L^{p,\infty}$.

   (b) If T is weak (p_0, q_0) and weak (p_1, q_1), then T is strong (p_θ, q_θ) with the usual reciprocal convexity **assuming** $p_\theta \le q_\theta$ (and of course $(q_0 \ne q_1)$.

1.5 Measure Theory

1.5.1 σ-algebras

Closed under complements, countable unions, and contain the empty set

1.5.2 Dynkin's π-λ theorem:

A π system is closed under finitely many intersections. A λ system (or Dynkin system) is contains $\varnothing, X$, is closed under complements, and closed under countable disjoint unions.

Theorem: if P is a π-system and $D \supseteq P$ is a Dunkin system, then $D \supseteq \sigma(P)$, the generated σ-algebra.

1.5.3 Measures

- An outer measure is 0 on the empty set, monotone, and countable subadditive.
- A measure is countable *additive* on disjoint sets. Can be defined on the σ-algebra of μ^* measurable sets for an outer measure μ^*: the sets with which you can "make change", i.e., F such that $\mu^*(E) = \mu^*(E \setminus F) + \mu^*(E \cap F)$ for *all* sets E
- Signed measures take at most one of $\pm\infty$, take 0 on the empty set, and are countable additive on disjoint subsets.

1.5.4 Decompositions

1. Lebesgue Decomposition

   For every two σ-finite measures μ and ν, there exists unique ν_s, ν_{ac} such that $\nu = \nu_s + \nu_{\mathrm{ac}}$ with $\nu_{\mathrm{ac}} \ll \mu$ and $\nu_s \perp \mu$.

   Recall that $a \ll b$ iff $b = 0 \implies a = 0$ and that $a \perp b$ iff there exists $A \sqcup B = X$ with $a(B) = 0$ and $b(A) = 0$. These measures are *absolute continuous* and *singular*, respectively.

   Proof: Let $\nu_s(E) = \sup \nu(E \cap N)$ over null sets N. Then let $\nu_{\mathrm{ac}}(E) = \sup \nu(E \cap M)$ over M ν_s-null. After this, the proof is elementary.

2. Hahn Decomposition

   There exists P and N totally positive and negative for a signed measures which does not take both ∞ and $-\infty$. Decomposition is *essentially* unique (i.e., up to null sets).

   Proof: suppose μ doesn't take $-\infty$. Show that there's a totally negative set inside everything with negative measure. Induct by soaking up more and more of the sup of the positive sets contained in side (induction is nested).

3. Jordan Decomposition

   Write a signed measure as $\mu^+ - \mu^-$ by intersecting with P and N from Hahn Decomposition.

4. Radon-Nikodym

   Let $\nu \ll \mu$ both be σ-finite. Then there is $f : X \to [0, \infty)$ measurable such that $\nu(A) = \int_A f \, d\mu$. We sometimes write $f = \frac{d\nu}{d\mu}$.

Proof: first for finite positive measures. Look at $\sup\{f \mid \int_A f\, d\mu \leq \nu(A)\}$. This set is closed under maxima, so take a max.

Let ν' be the difference $\nu(A) = \int_A g\, d\mu$. If this is non-zero, then $\nu'(X) > \varepsilon\mu(X)$ for some ε because μ is finite. Pull a Hahn decomposition on the difference and consider an indicator function on the positive part, adding it to g (the sup).

Then prove for σ-finite positive measures, then signed and complex measures in the standard way.

1.5.5 Bounded Variation

1. Total Variation

   The total variation of a signed measure μ is $(\mu^+ + \mu^-)(X) = |\mu|(X) =: \|\mu\|$.

2. BV Functions on $\mathbb{R}$

   A function $f : \mathbb{R} \to \mathbb{C}$ is of bounded variation if $\sup \sum |f(x_k) - f(x_{k-1})| < \infty$ where the supremum is taken over all finite partitions.

   Theorem: f is BV and real-valued iff it is the difference of two bounded increasing functions. Furthermore, left/right limits exist everywhere (even at ∞). As a corollary, BV functions have at most countably many discontinuities.

   Proof: let T_f be the total variation of f (the variation from $-\infty$ to x). Then $T_f + f$ and $T_f - f$ are the two functions whose difference is $2f$ and are both bounded increasing functions (takes some proof).

3. Normalized BV

   A function is NBV is its limit at $-\infty$ is 0, it is right-continuous, and BV.

   Theorem: f is NBV iff $f = F_\mu$ where $F_\mu(x) = \mu(-\infty, x]$ for some complex Borel measure μ.

1.5.6 Riesz Representation (for $C(X)$)

If X is a locally convex Hausdorff space, $I : C_c(X) \to \mathbb{C}$ is a positive linear functional (i.e., positive on positive functions), then there is a unique Radon (inner regular on open sets, outer regular on Borel sets, finite on compact sets) measure μ such that

$$I(f) = \int f \, d\mu$$

Proof (uniqueness): if the measure exists, it's uniquely determined on open sets by I. Pick U open, and for arbitrary $K \subseteq U$ compact, choose $K \prec f \prec U$. Then take a sup.

Then apply regularity for uniqueness of μ.

(existence): Define using the above sup. This gives a pre-measure, which can be used to create an outer measure as the infimum over partitions into oen sets. Show outer regularity and the measurability of Borel sets. An involved proof, but not too hard.

There are related versions. We list a short form for each statement below.

1. Realizations of Riesz Representation

   (a) positive functionals on $C_c(X)$ = positive Radon measures

   (b) $C_0(X)^* = \{\text{regular complex Borel measures}\}$ where C_0 is functions vanishing at infinity

   (c) If X is compact, then $M(X) \simeq C(X)^*$ where $M(X)$ is the set of all complex Radon measures

   Note: finite Borel measures on (separable?) metric spaces are always regular. Proof: consider the sets where regularity holds and show it's a σ-algebra at least as big as the Borel σ-algebra.

1.5.7 Portmanteau Theorem

If S is a metric space, $\mu_n \rightharpoonup \mu$ probability measures converge weakly-star to the *finite* positive measure μ if and only if any of the following conditions holds. (We write $\mathbb{E}_n[f] = \int f \, d\mu_n$):

1. $\int f \, d\mu_n \to \int f \, d\mu$ for all bounded continuous f
2. $\mathbb{E}_n[f] \to \mathbb{E}[f]$ for all bounded Lipschitz f
3. $\limsup \mathbb{E}_n[f] \leq \mathbb{E}[f]$ for all USC f bounded above
4. $\liminf \mathbb{E}_n[f] \geq \mathbb{E}[f]$ for all LSC f bounded below
5. $\limsup \mu_n(C) \leq \mu(C)$ for all closed C

6. $\liminf \mu_n(U) \geq \mu(U)$ for all open U
7. $\lim \mu_n(A) = \mu(A)$ for all *continuity sets* A of μ, i.e., sets where the boundary has total variation 0.

1.5.8 Hausdorff measure

$$H_{p,\delta}(A) \coloneqq \inf\{\sum (\operatorname{diam} B_j)^p \mid A \subseteq \bigcup_{\mathbb{N}} B_j, \operatorname{diam} B_j \leq \delta\}$$

Then $H_{p,\delta}$ is monotone decreasing in δ, so $\lim_{\delta\to 0} H_{p,\delta} = \sup_{\delta>0} H_{p,\delta} =: H_p$.

This is a metric outer measure (hence Borel σ-algebra is measurable), and equals Lebesgue measure when dimensions are the same up to some constant: $m_d = H_d$.

Given a Borel set, $\dim(S) = \inf\{d \geq 0 \mid H_d(S) = 0\} = \sup\{d \geq 0 \mid H_d(S) = \infty\}$.

1.6 Convergence and Integration Results

1.6.1 Types of Convergence

1. AE — pointwise almost everywhere
2. L^1 norm convergence
3. Weak L^1 — $\int f_n g \to \int fg$ for all $g \in L^\infty \simeq (L^1)^*$
4. L^∞ — $\forall \varepsilon > 0$, $|f_n(x) - f(x)| \leq \varepsilon$ is eventually true almost everywhere
5. AU — almost uniformly, $\forall \varepsilon > 0$ there exists $\mu(E) \leq \varepsilon$ such that $f_n \to f$ uniformly on E^c
6. M — convergence (globally) in measure, $\forall \varepsilon > 0$, $\mu\{|f - f_n| > \varepsilon\} \to 0$.

We have the following diagram:

$$\begin{tikzcd}
L^\infty \arrow[r] & AU \arrow[r, shift right] \arrow[dr, squiggly, blue, "bc"] & M \arrow[l, dashed, shift right] & \\
& AE \arrow[ul, squiggly, blue, "ac"] \arrow[u, squiggly, blue, "a"] \arrow[r, squiggly, shift left, "a"] & L^1 \arrow[u] \arrow[ul, dashed] \arrow[l, dashed, shift left] \arrow[r] & \text{weak } L^1
\end{tikzcd}$$

This diagram is incomplete, but note that ae implies L^1 if $\|f_n\| \to \|f\|$ by a clever Fatou argument: consider $|f_n - f| \leq |f_n| + |f|$ so apply Fatou to $|f_n| + |f| - |f_n - f|$.

In a finite measure space, AE and AU are equivalent.

If the f_n are dominated by an L^1 function, then AE implies L^1, AE and AU are equivalent, and M implies L^1.

We have the following counter-examples to convergence implications:

1. $\chi_{[n,n+1]}$ goes to zero ptwsie but not unif, in L^∞, in L^1, almost-unif, or in measure

2. $\frac{1}{n}\chi_{[0,n]}$ goes to zero uniformly (hence ptwsie, L^∞, almost-unif, and in measure), but not L^1

3. $n\chi_{[1/n,2/n]}$ goes to zero ptwise, almost unif (thus in measure), but not uniformly, nor in L^∞, nor in L^1.

4. typewriter sequence goes to zero in measure and L^1, but not ptwise ae (nor almost-unif, nor L^∞, nor unif).

1.6.2 Monotone Convergence Theorem

A monotone increasing sequence of functions satisfies $\int \lim f_n = \lim \int f_n$.

Proof (from baby Rudin): Assume $f \geq 0$. Let $0 < c < 1$ and $0 \leq s \leq f$ simple. Set $E_n = \{x \mid f_n(x) \geq cx(x)\}$. Then $E = \bigcup E_n$ and

$$\int_E f_n \geq \int_{E_n} f_n \geq c \int_{E_n} s$$

and so let $n \to \infty$ then $c \to 1$. Other direction is trivial.

1.6.3 Fatou's Lemma

$\int \liminf f_n \leq \liminf \int f_n$. Proof: apply monotone convergence

1.6.4 Lebesgue Dominated Convergence Theorem

If $f_n \leq g$ and $\int g < \infty$, then $\lim \int f_n = \int \lim f_n$.

Proof: take a difference and apply Fatou.

1.6.5 Vitali Convergence Theorem (characterization of L^p convergence)

Let $f_n, f \subseteq L^p$ with $1 \leq p < \infty$. Then $f_n \to f$ in L^p if and only if:

1. **(Measure convergence)** f_n converge in measure to f, i.e., $\forall \varepsilon > 0, m\{|f - f_n| \geq \varepsilon\} \to 0$.
2. **(Tight)** For every $\varepsilon > 0$ there exists $\mu(E) < \infty$ such that $\int_{E^c} |f_n|^p < \varepsilon$.
3. **(Uniformly integrable)** For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\mu(E) < \delta$ implies $\int_E |f_n|^p < \varepsilon$.

Note that on a finite measure space, the second condition follows from the third.

Proof: not important.

1.6.6 Ptwise a.e. and $||f_n|| \to ||f||$ implies $f_n \to f$

If $f_n \to f$ pointwise a.e. and $\|f_n\| \to \|f\|$, then $f_n \to f$ in L^1.

Proof: We have $|f_n| + |f| - |f_n - f| \geq 0$ and so we can apply Fatou. Everything cancels nicely.

1.6.7 Egoroff / Egorov's Theorem

On a finite measure space, $f_n \to f$ ptwise ae implies almost uniform convergence, i.e., $\forall \varepsilon > 0$ there exist $|E| < \varepsilon$ outside of which convergence is uniform

Proof: $E_{n,k} = \bigcup_{m \geq n} \{|f_m - f| \geq 1/k\}$. For each k, pick n_k such that $\mu(E_{n_k,k}) < \varepsilon/2^k$. Now $B = \bigcup_k E_{n_k,k}$ is the bad set which is small.

1.6.8 Lusin's Theorem

Let X be a space with a Radon measure and Y a second-countable topological space. For $f : X \to Y$ measurable, outside an arbitrarily small exceptional set, f is continuous.

Proof: Egorov's theorem and the density of smooth (or simply continuous) functions. Take continuous functions converging pointwise ae, then continuity is preserved by the uniform convergence outside exceptional sets

1.6.9 Tonelli's Theorem

If $f \geq 0$ measurable, X, Y are σ-finite measure spaces, then we can swap the order of integration (where $X \times Y$ is given the product measure.)

1.6.10 Fubini's Theorem

If X, Y are σ-finite and f is $X \times Y$ integrable, then we may swap the order of integration.

Often use Tonelli and then Fubini.

1.6.11 Change of Variables / Transformation Formula

For $T \in \mathrm{GL}(n)$, $\int f = \left|\det T\right| \int f \circ T$.

Stronger version: Ω open and $G : \Omega \to \mathbb{R}^n$ is a C^1 diffeomorphism (onto its imagE). If f is measurable, then $f \circ G$ is, and if $f \geq 0$ or $f \in L^1$, we have:

$$\int_{G(\Omega)} f = \int_{\Omega} f \circ G \left|\det DG\right|$$

where $\left|\det DG\right|$ is the absolute value of the determinant of the Jacobian matrix (the matrix of partials).

1.7 Hardy-Littlewood Maximal Function

1.7.1 Covering Lemmas

1. Vitali Covering Lemma (Finite) Let $B_1, \ldots, B_n \subseteq \mathbb{R}^d$ be a collection of balls. Then there is a disjoint subcollection which (when tripled) covers the original union of balls.

   Proof: First, meet every ball starting with the largest ball by then selecting the maximally sized disjoint ball inductively. Then triangle inequality.

2. Vitali Covering Lemma (Infinite) If $\{B_j\} \subseteq \mathbb{R}^d$ is a collection of balls with $\sup \mathrm{radius}(B_j) < \infty$, then there is a *countable* subcollection whose *quintuples* cover the original union. Furthermore, every ball meets one of the subcollection balls.

   Proof: Apply the finite version to balls whose radius is between $1/2^n$ and $1/2^{n+1}$ of the sup. For each n take the maximal disjoint subcollection of balls (of the given size) disjoint from the first n many sets chosen.

   Alternatively, I think there's a proof involving taking balls whose radius is 2/3 of the sup and inducting as before.

1.7.2 Weak L^1 bound

The Hardy-Littlewood maximal functions is $M(f) = \sup_{r>0} \fint_{B(x,r)} |f|$, defined when $f \in L^1_{\text{loc}}$.

M is weak type $(1,1)$. To show this, we need to argue $m\{Mf > \alpha\} \lesssim \frac{\|f\|_{L^1}}{\alpha}$ for all $\alpha > 0$.

If $x \in \{Mf > \alpha\}$, there is some ball around x witnessing $\fint_{B(x,r_x)} |f| > \alpha$.

By the infinite Vitali Covering Lemma (rephrased as follows): we can take a countable disjoint family of balls that cover $c/3^n$ of the volume of $\{Mf > \alpha\}$ for any $c < m\{Mf > \alpha\}$. The estimate trivially follows.

1.7.3 L^p boundedness

M is trivially bounded in L^∞ (since $\|Mf\|_{L^\infty} \leq \|f\|_{L^\infty}$). Thus by Marcinkiewicz, M is bounded in L^p.

Alternatively, $f = f\chi_{|f|>t/2} + f\chi_{|f|<t/2} = g + h$ and apply sublinearity. Sinee $Mh \ //get/2$, we need only worry about the former being large. Rephrasing using the distributional integral form and the weak L^1 estimate:

$$\begin{aligned}
\|Mf\|_{L^p}^p &= \int pt^{p-1} m\{Mf > t\}\, dt \\
&\leq \int pt^{p-t} m\{Mg > t\}\, dt \\
&\leq \int p \frac{2C}{t} \int_{|f|>t/2} |f|\, dx\, dt \sim \|f\|_{L^p}^p
\end{aligned}$$

1.7.4 Lebesgue Differentiation

For every Lebesgue point (these are ae point) of some $f \in L^1_{\text{loc}}$, averages on "nicely shrinking sets" of f converge to f. Nicely shrinking means $E_r \subseteq B(x,r)$ and $|E_r| > \alpha |B(x,r)|$.

Proof: Shrinking nicely makes no difference. Instead, let's just consider the result for balls. We'll show that almost every point is a Lebesgue point, i.e., where $\fint_{B(x,r)} f \to f(x)$ as $r \to 0$.

It suffices to assume $f \in L^1$ by restricting to a ball. Find $g \in C_c$ such that $\|f - g\|_{L^1} \leq \varepsilon$. Then

$$\limsup \left| \fint f - f \right| \leq M(f-g) + 0 + |f - g|$$

The weak type $(1, 1)$ estimate on M finishes the result by considering the set where the former is bigger than α (contained in the union of when each latter term is bigger than $\alpha/2$). Apply Chebyshev and the weak $(1, 1)$ estimate.

We can strengthen $⨍ (f(y) - f(x))\, dy \to 0$ ae to $⨍ |f(y) - f(x)|\, dy \to 0$ ae:

Apply the above result to $|f(x) - c|$ for a countable dense set of constants c. Take the union of their corresponding null sets.

So in fact, absolute values of averages (minus the function) on nicely shrinking sets goes to zero!

1.8 Useful Results to Remember and Repeated Problems

1.8.1 Special Integrals to Memorize

1. Integral of a Gaussian

$$\int_{-\infty}^{\infty} e^{-ax^2} = \frac{\sqrt{\pi}}{\sqrt{a}}$$

   Proof:

$$\begin{aligned}\left(\int e^{-x^2}\, dx\right)^2 &= \int_{\mathbb{R}^2} e^{-r^2} r\, dr\, d\theta \\ &= 2\pi \int_0^{\infty} re^{-r^2}\, dr\end{aligned}$$

   and apply a change of coordinates to get π.

2. The Gamma Function

$$\begin{aligned}\Gamma(z) &= \int_0^{\infty} x^{z-1} e^{-x}\, dx \\ &= \int_0^{\infty} x^z e^{-x} \frac{1}{x}\, dx\end{aligned}$$

   Note that $\Gamma(n) = (n-1)!$. To prove this, describe a recurrence relation with integration by parts, and show $\Gamma(1) = 1 = 0!$.

   To find $\Gamma(1/2)$, apply a change of coordinates and use the Gaussian integral.

3. The Beta Function

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$$

for $\operatorname{Re} x, \operatorname{Re} y > 0$.

Seems extremely unlikely to appear, but usefully, $B(x, y) = B(y, x)$ and more usefully:

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}$$

1.8.2 Integration Techniques

1. Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

2. Trig substitution

   (a) If an integral contains $a^2 - x^2$ use $x = a\sin\theta$ and $dx = a\cos\theta\,d\theta$.
   Example:

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \int \frac{a\cos\theta}{a\cos\theta}\,d\theta$$

   (b) If an integral contains $a^2 + x^2$ use $x = a\tan\theta$ and $dx = a\sec^2\theta$.
   Example:

$$\begin{aligned}\int \frac{1}{x^2 + a^2}\,dx &= \int \frac{a\sec^2\theta}{a^2 + a^2\tan^2\theta}\,d\theta \\ &= \frac{1}{a}\,d\theta = \frac{1}{a}\arctan(x/a)\end{aligned}$$

   (c) If an integral contains $x^2 - a^2$, try $x = a\sec\theta$ and $dx = a\sec\theta\tan\theta\,d\theta$.
   Example:

$$\begin{aligned}\int \sqrt{x^2 - a^2}\,dx &= a^2 \int \tan\theta\tan\theta\sec\theta \\ &= a^2 \int \tan^2\theta\sec\theta\,d\theta \\ &= a^2 \int \sec^3 - \sec\ d\theta = \ldots\end{aligned}$$

3. Hyperbolic trig substitution

(a) With $\sqrt{a^2+x^2}$ try $x = a\sinh u$ and $dx = a\cosh u\, du$.

This is an alternative to the $x = a\tan\theta$ substitution.

(b) With $\sqrt{x^2-a^2}$ try $x = a\cosh u$ and $dx = a\sinh u\, du$.

This is an alternative to the $x = a\sec\theta$ substitution.

4. Partial fractions

   Write $f/\prod g_i = A/g_1 + B/g_2 + \cdots$. Easy to solve, and allows for integrating messy rational functions when all you know how to do is log.

5. Half tangent substitution

   This is a very sneaky substitution used in dealing with rational functions of sin and cos.

   Write $t = \tan(x/2)$ and then

$$\begin{aligned}\sin x &= \frac{2t}{1+t^2}\\ \cos x &= \frac{1-t^2}{1+t^2}\\ \frac{dt}{dx} &= \frac{1+t^2}{2}\end{aligned}$$

   Examples:

$$\begin{aligned}\int \csc x\, dx &= \int \frac{1}{\sin x}\, dx\\ &= \frac{1+t^2}{2t}\frac{2}{1+t^2}\, dt\\ &= \log t = \log\tan(x/2)\end{aligned}$$

   Another example:

$$\int_0^{2\pi} \frac{1}{2+\cos x}\, dx = \int \frac{2}{3+t^2}\, dt$$

1.8.3 Continuous Projections onto Finite Dimensional Subspaces

Let $V \subseteq W$ be a finite dimensional subspace of the Banach space W. Show there is a continuous projection $P : W \to V$.

Proof: Write $V = \operatorname{span}\{x_1, \ldots, x_n\}$. For each i, consider $p_i : V \to V$ via $p_i(a_j x_j) = a_i$. Extend by Hahn–Banach.

1.8.4 L^∞ bounded by L^2 norm on subspace, hence finite dimensional

Suppose $\{\phi_n\}$ is an ON set of continuous functions in $L^2([0,1])$ and $S = \overline{\{\phi_n\}}$. Suppose $\sup_{f\in S\setminus\{0\}} \|f\|_{L^\infty}/\|f\|_{L^2} < \infty$. Show that S is finite dimensional.

Proof: Define $E_x : S \to \mathbb{R}$ via $E_x(f) = f(x)$. This is a bounded linear functional on S, so $E_x(f) = \langle f, g_x\rangle$. Note that $\|g_x\|_{L^2}^2 = |g_x(x)| \le M\|g_x\|_{L^2}$.

Finally, Bessel's inequality gives us

$$M^2 \ge \|g_x\|_{L^2}^2 \ge \sum_n |\langle \phi_n, g_x\rangle|^2 = \sum |\phi_n(x)|^2$$

so by integrating in x, we get $\dim S = \#\{\phi_n\} \le M^2 < \infty$.

1.8.5 Weak Convergence in ℓ^1 is Strong Convergence

Suppose $x^n \rightharpoonup 0$. Let's show $\|x^n\| \to 0$. Suppose $\|x^n\| \ge 1$ for some ε. We'll show there's a subsequence $\{x^{n_k}\}$ with $\langle y, x^{n_k}\rangle > \varepsilon$ for some $y \in \ell^\infty$.

Let n_0 be such that $\sum_{n\le n_0} \left|x_n^0\right| \ge 1/3$. Then For $0 \le k \le n_0$, take $y_k = \operatorname{sign} x_k^0$.

Then $\left\langle y, x^0 \right\rangle > 1/3$.

Next, pick k_1 such that $\sum_{n\le n_0} \left|x_n^{k_1}\right| < 1/6$ and $\sum_{n_0\le n\le n_1} \left|x_n^{k_1}\right| \ge 1/3$.

Then for $n_0 < j \le n_1$, take $y_j = \operatorname{sign} x_j^{k_1}$. Note that $\left\langle y, x^0\right\rangle > 1/3$ and $\left\langle y, x^{k_1}\right\rangle > 1/3$.

Inducting, we get $\left\langle y, x^{k_j}\right\rangle > 1/3$ for all j, a contradiction to weak convergence to zero.

Suppose $x^n \rightharpoonup x$. Then $x^n - x \rightharpoonup 0$, so $\|x^n - x\| \to 0$.

1.8.6 Borel σ-algebra on $C([0,1])$ characterization

Let $X = C([0,1])$. Show that the Borel σ-algebra on X is the smallest σ-algebra that contains all sets of the form

$$S(t,B) = \{f \in X \mid f(t) \in B\}$$

for B Borel and $t \in [0,1]$. Alternatively, this question can be phrased in terms of measurability of the evaluation maps.

Proof hint: consider $t \in \mathbb{Q} \cap [0,1]$.

1.8.7 Non-decreasing functions take Borel sets to Borel sets

Proof idea: Consider $\mathcal{F} = \{A \subseteq \mathbb{R} \mid f(A) \text{ is Borel}\}$. Show it's a σ-algebra.

1.8.8 Nowhere-zero Fourier transform implies translates are L^2 dense

Let $f \in L^2$ and assume $|\hat{f}| > 0$ a.e. Show that $f_y(x) = f(x-y)$ is norm dense in $L^2(\mathbb{R})$.

Proof: Consider $M = \overline{\operatorname{span}\{f_a\}_{a\in\mathbb{R}}}$. Suppose $M \neq L^2$. Then there is $0 \neq g \in M^\perp$. Apply Plancherel:

$$\begin{aligned}\int \hat{f}_a \hat{g} &= \int e^{-2\pi i a\xi} \hat{f}\hat{g} \\ &= \widehat{\hat{f}\hat{g}}(a) = 0\end{aligned}$$

where $\hat{f}\hat{g} \in L^1$. Applying Fourier inversion gives us $\hat{f}\hat{g} = 0$ everywhere.

1.8.9 L^2 bounded and almost everywhere to zero implies weakly to zero

2 Complex Analysis

2.1 Introductory Theorems and Cauchy's Integral Formula

2.1.1 Cauchy-Riemann Equations

For $f = u + iv$, we have that f is holomorphic implies

$$\begin{cases} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{cases}$$

Proof: apply complex differentiability along vertical and horizontal lines.

Furthermore, if u and v are real differentiable, then $f = u + iv$ is complex differentiable if and only if u and v satisfy the Cauchy-Riemann equations.

Proof: consider Morera's Theorem. To show that f has integral zero on a closed loop, apply Green's Theorem. We start with

$$\int_\gamma (u+iv)(dx+idy) = \int_\gamma udx - vdy + i\int_\gamma vdx + udy$$

but both terms die by Green's Theorem.

2.1.2 Goursat's Theorem

The integral of a holomorphic function around a rectangle (or triangle) is zero.

Proof: decompose your rectangle (or triangle) into 4^n pieces with $1/2^{\mathrm{n}}$ the side-length and then approximate f by its linearization: $|f(z) - f(z_0) - f'(z_0)(z-z_0)| \leq \varepsilon$ for $|z - z_0| < r$.

Pick a fine enough decomposition so that this holds on one of your subshapes which has at least the average itnegral and then take a sup approximation of the integral.

2.1.3 Cauchy Integral Theorem

The integral of a holomoprhic f on cycles homologous to zero (or simpler: in a simply connected domain) is zero.

Proof: relies on Green's theorem:

$$\int_\gamma u\,dx - v\,dy = \iint \partial_x v - \partial_y u$$

Writing $f = u + iv$ and $dz = dx + idy$, breaking the integral into real and imaginary parts, and applying the Cauchy-Riemann equations is enough.

2.1.4 Cauchy's Integral Formula

When γ is a curve in a simply connected region where f is analytic and winds once around a,

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz$$
$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz,$$

where the latter is sometimes called Cauchy's differentiation formula.

Note that winding multiple times just multiplies the integral's value by that amount.

Proof: Note that $\frac{f(z)-f(a)}{z-a}$ has a removable singularity at a. Technically all we need is that Cauchy's Theorem holds on loops around a.

Rewriting gives us the result.

A proof that Cauchy's Theorem hold here: take an arbitrarily small circle and assume wlog $a = 0$.

$$\begin{aligned}\left|\frac{1}{2\pi i}\int \frac{f(z)}{z}\,dz - f(0)\right| &= \left|\frac{1}{2\pi i}\int \frac{f(z)-f(0)}{z}\,dz\right| \\ &\leq \int_0^{2\pi} \frac{|f(z(t)) - f(0)|}{\varepsilon}\varepsilon\,dt \to 0\end{aligned}$$

All we need for this is that $\int \frac{1}{z} = 2\pi i$ around the origin. This is computational.

For Cauchy's differentiation formula, write $f'(z) = \lim_{w\to z} \frac{f(w)-f(z)}{w-z}$ and apply Cauchy's integral formula to the numerator. Swapping limits and integrals by uniform convegence gives us the result. Then induct.

1. Pompeiu's Formula

   For D a bounded domain with piecewise smooth boundary, and g an $\mathbb{R}^2$-smooth complex-valued function up to the boundary, then

$$g(w) = \frac{1}{2\pi i}\int_{\partial D} \frac{g(z)}{z-w}\,dz - \frac{1}{\pi}\iint_D \frac{\partial g}{\partial \bar{z}}\frac{1}{z-w}\,dx\,dy$$

   This is proven with Green's Theorem and works when g is not holomorphic. See Gamelin's *Complex Analysis*, page 127.

2. Bergman space

   A Bergman space is L^p intersected with holomorphic functions. Typically, qual problems will ask about $A^2(\mathbb{D})$. We have the explicit orthonormal basis

$$\left\{ e_n := \sqrt{\frac{n+1}{\pi}} z^n \right\}.$$

   Orthonormality is easy. To show these span, consider the following calculation. We compute

$$\begin{aligned}
\langle f, e_n\rangle &= \sqrt{\frac{n+1}{\pi}} \int_{\mathbb{D}} f(re^{i\theta}) r^{n+1} e^{-in\theta}\,dr\,d\theta. \\
&= \sqrt{\frac{n+1}{\pi}} \int_{\mathbb{D}} r^{2n+1} \frac{f(re^{i\theta})}{r^{n+1}e^{i(n+1)\theta}} re^{i\theta}\,d\theta\,dr \\
&= \sqrt{\frac{n+1}{\pi}} \int_0^1 r^{2n+1} f^{(n)}(0)\,dr \\
&= c_n f^{(n)}(0)
\end{aligned}$$

   by Cauchy's diffentiation formula.

   Interestingly, the evaluation map is bounded in this space. To show $L_z : f \mapsto f(z)$ is bounded, we apply the mean value property (it's trivial after this).

   In fact, $L_z(f) = \langle f, g_z\rangle$, where

$$g_z(w) = \frac{1}{\pi(1-\bar{z}w)^2}$$

2.1.5 Liouville's Theorem

A bounded entire function is constant.

Proof: Write $f = \sum a_k z^k$. Then Cauchy's integral formula gives

$$a_k = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta^{k+1}}\, d\zeta$$

so $|a_k| \leq M/r^k$. Send $r \to \infty$. Note also that $a_k = f^{(k)}(0)/k!$.

2.1.6 Morera's Theorem

If f is continuous and has integral zero on all cycles in a region, f is analytic.

Proof: construct an anti-derivative F. Then show that $F' = f$. Since F is holomorphic, it's analytic (and smooth).

1. A Standard Consequence of Morera

   If holomorphic $f_n \to f$ uniformly, then $\int_\gamma f = \lim \int_\gamma f_n = 0$ on closed curves, so f is holomorphic too.

2.1.7 Residue Theorem

$$\frac{1}{2\pi i} \int_\gamma f(z)\, dz = \sum_j \mathrm{Res}_{z=a_j} f(z)$$

if f is analytic around γ except at the poles a_j.

As a reminder, if f has a simple pole at a, then

$$\mathrm{Res}_a f = \lim_{z \to a} (z-a) f(z)$$

If f has a pole of order n at a, then

$$\mathrm{Res}_a f = \lim_{z \to a} \left(\frac{(z-a)^n}{(n-1)!} f(z) \right)^{(n-1)}$$

where the $(n-1)$ exponent indicates $n-1$ many derivatives.

2.1.8 Argument Principle

The integral $\int_\gamma \frac{f'}{f} = 2\pi i(Z - P)$ where Z is the number of zeroes and P is the number of poles (counted by winding number as well as multiplicity).

Note:

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = \int_{f(\gamma)} \frac{1}{w}\,dw$$

Proof: Write $f = \prod(z - a_i)^{k_i} g(z)$ where g lacks zeroes or poles. For simplicity, consider $f = z^k g$. Then

$$\frac{f'}{f}(z) = \frac{k}{z} + \frac{g'}{g}(z)$$

Thus the residue of f'/f at 0 is k. Do the same thing for poles and get a minus sign.

2.1.9 Schwarz Lemma

If f is analytic in $\mathbb{D}$, bounded by 1, and $f(0) = 0$, then $|f(z)| \le |z|$ and $|f'(0)| \le 1$. Equality anywhere implies $f(z) = e^{i\theta} z$.

Proof: apply the maximum principle to $f(z)/z$ on balls of radius $r < 1$, where f/z is extended by $f'(0)$ at the origin.

Every $z \in B(0,1)$ is bounded: $|g(z)| \le |g(z_r)| \le \frac{1}{r}$ for some $z_r \in \partial B(0,r)$. Send $r \to 1$.

Suppose $|f(z)| = |z|$ for some $z \neq 0$. Then use the correct radius and deduce g is constant.

2.1.10 Schwarz-Pick lemma

The same result as the Schwarz Lemma, but after applying an automorphism of the disk to relabel the origin (on both sides of the function).

2.1.11 Rouché's Theorem

Let γ be homologous to zero in Ω such that $n(\gamma, z)$ is 0 or 1 for any point z not on γ, i.e., it doesn't wind more than once around any point.

Suppose f, g are analytic on Ω and $|f - g| < |f|$ on γ. Then f and g have the same number of zeroes enclosed by γ.

Proof: First note that f, g are zero-free on γ. Then

$$\left|\frac{g}{f} - 1\right| < 1$$

so $F = g/f$ is contained in $B(1, 1)$ on γ, and so $n(\Gamma, 0) = 0$ when $\Gamma = F \circ \gamma$. Since $n(\Gamma, 0) = \frac{1}{2\pi i} \int \frac{F'}{F}$ counts the number of zeroes minus poles by the argument principle, we're done.

2.1.12 Symmetrized Rouché's Theorem

This version is stronger and thus is the one to remember.

Suppose $|f - g| < |f| + |g|$ on ∂K holds strictly. Then f and g have the same number of roots in K.

Understanding: by replacing g with $-g$, we get that $|f + g| \neq |f| + |g|$ on the boundary, i.e., $\arg f \neq \arg g$. Thus f and g contain the same number of zeroes if they never point in the same direction.

Proof: not necessary, but can be found on wikipedia.

2.1.13 Open Mapping Theorem

If $f : U \to \mathbb{C}$ is non-constant and holomorphic, it's an open map (i.e., open sets are taken to open sets).

Proof: Suppose $f(0) = 0$. Then taking a small circle C_ε and $f(C_\varepsilon) = \Gamma$, we note that Γ winds exactly once around an entire disc of points around 0. Thus, since $n(\Gamma, a) = n(C_\varepsilon, z)$ we get $f(z) = a$ solved exactly once.

Alternatively, Rouche's theorem can be used, or the maximum modulus principle.

2.1.14 Removable Singularities and Riemann's Theorem

If a holomorphic function is defined in a punctured neighborhood of a, the following are equivalent:

1. There is a holomorphic extension over a
2. There is a continuous extension
3. There's a neighborhood of a on which f is bounded
4. $\lim_{z\to a}(z - a)f(z) = 0$

Proof: consider $(z - a)^2 f(z)$ and note that it's holomorphic. We can get a power series for f.

2.2 Normality and Uniform Convergence on Compact Sets

2.2.1 Definitions

1. A family of holomorphic functions is normal if every sequence of functions has a subsequence that converges uniformly on compact sets.

   This is equivalent to being a pre-compact subset of $C(\Omega; \mathbb{C})$ with the metric that yields uniform convergence on compact sets (take sup-norms on compact sets which exhaust your domain and take their sum (with 2^{-n}) in the usual way).

   We can also consider our codomain to be $\mathbb{C}^*$, the Riemann sphere with the spherical metric. This gives the following definition:

2. A family of meromorphic functions is normal if every sequence of functions has a subsequence that converges uniformly in the spherical metric on compact sets.

   This is equivalent to the following description when applied to holomorphic functions:

3. A family of holomorphic functions is normal if every sequence contains either a subsequence that converges uniformly on every compact set or a subsequence that tends uniformly to ∞ on every compact set. This is sometimes called *normal in the classical sense*.

   Note that when meromorphic functions converge uniformly on compact sets, the limit is meromorphic (and no new zeroes are introduced), so viewed as meromorphic functions, Definition 3 is exactly what's expected.

2.2.2 Montel's Theorem

1. A family of holomorphic functions defined on an open subset is normal if and only if it is (locally) uniformly bounded. (See Ahlfors p224.)

   Proof: We prove equicontinuity with Cauchy's Integral Formula:

   $$f(z) - f(w) = \frac{1}{2\pi i} \int_\gamma f(\zeta) \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right) d\zeta$$

   Bound the integrand to get equi-Lipschitz.

2. A family of holomorphic functions defined an open set is normal if they all omit the same two values.

Proof: using the modular λ function, a universal cover of $\mathbb{C} \setminus \{0, 1\}$ by the upper half-plane, which is invariant under the *congruence subgroup*: 2×2 matrices which are congruent to I mod 2. Note that λ is conformal from the fundamental region (bounded by $\mathrm{Re}(z) = 0$, $\mathrm{Re}(z) = 1$, and the circle $\partial B(1/2, 1/2)$).

We apply the Monodromy Theorem to lift a function that avoids 0 and 1 to the upper-half plane, then conformally map to a disc to get them all bounded. This is the proof of Picard's Little Theorem, too.

This is also called the *fundamental normality test*.

2.2.3 Hurwitz's Theorem

Let $\{f_n\}$ be holomorphic in a region and converge locally uniformly to f (which isn't constantly zero).

If f has a zero of order m at a, then for every radius $r > 0$, eventually the functions f_n all have exactly m zeroes up to multiplicity in $B(a, r)$. Furthermore, these zeroes converge to a as $n \to \infty$.

Proof: Make sure that $|f|$ is bigger than some $\delta > 0$ on the circle. Then $(f_n'/f_n) \to (f'/f)$ uniformly on the circle and we can apply the argument principle.

Alternatively, if each f_n has at most m zeroes, then the limit must either be identically zero or have also at most m zeroes. This is a convenient way to phrase it sometimes.

Remark: cf. Harnack's Principle

2.2.4 Marty's Theorem

A family of meromorphic (including holomorphic) functions f is normal *in the classical sense* (i.e., normal as functions into $\mathbb{C}^*$ with respect to the spherical metric) if and only if the spherical derivatives are locally bounded.

The spherical derivatives are

$$\rho(f) = \frac{2|f'(z)|}{1 + |f(z)|^2}$$

Proof: Interpret the spherical derivative geometrically as $d(f(z_1), f(z_2))$.

2.3 Analytic Extensions/Continuations

2.3.1 Monodromy Theorem

Analytic extensions from the same intial germ along homotopic arcs yields the same terminal germ.

Reminder on notation: a germ is an analytic function in a small disk about a point (really just a power series at a point itself, since germs are the same if they agree in some shared neighborhood.

An analytic extension is a finite collection of balls along a path along with analytic functions in each ball that agree where they overlap. *Note: these are functions on the balls themselves and are not to be viewed as functions from the complex plane.* Reason for warning: the path can cross over itself and take different values "later on".

2.3.2 Picard's Little Theorem

If f is entire and non-consatnt, then f can omit at most one point.

Proof: pullback with the Monodromy Theorem via λ to the upper half-plane, then map to the disc and apply Liouville's Theorem.

Note: e^z is why this can't be strengthened.

2.3.3 Picard's Great Theorem

If f has an essential singularity at a point, then on any punctured neighborhood, f takes on all complex values (with at most one exception) infinitely often (clearly).

Proof: Suppose f omits 0 and 1. Then $\{f(\varepsilon_n z)\}$ is normal on $\mathbb{C} \setminus \{0\}$ by Montel (version 2), since they all avoid 0 and 1 (here $\varepsilon_n \to 0$).

The limit is either constantly ∞ or holomorphic.

1. If the limit is ∞, apply this argument to $1/f(\varepsilon_n z)$ and deduce that f is meromorphic.

2. If the limit is some holomorphic g, we have that f must be bounded near 0 and hence has a removable singularity.

Note: $e^{1/z}$ is why this can't be strengthened.

Furthermore, Picard's Little Theorem is a corollary by considering $f(1/z)$.

2.4 Maximum Principle

2.4.1 The Maximum Principle

A non-constant holomorphic function has no maximum of $|f|$ in the interior of a region.

Similarly, a harmonic function has neither a maximum nor minimum in the interior of a region.

2.4.2 Hadamard's 3 lines Theorem

Let f be bounded and holomorphic on the strip between $\mathrm{Re}(z) = 0$ and 1. Let $M(x) = \sup_y |f(x+iy)|$. Then $\log M(x)$ is convex, i.e.,

$$M(x) \leq M(0)^t M(1)^{1-t}$$

where $x = 0t + 1(1-t) = 1-t$.

Proof: define $F(z) = f(z)M(0)^{1-z}M(1)^z$. Then $|F(z)| \leq 1$ on the boundary of the strip. We would like to apply the maximum principle, since if we can show it's bounded by 1 in the interior of the strip, we're done (since the complex part of the exponential won't affect the magnitude).

Write $F_n = Fe^{z^2/n}e^{-1/n}$. These tend to F pointwise and all go to zero as $|z| \to \infty$, so they're bounded by 1 on the interior of the strip (since they're bounded by 1 on the boundary and applying the maximum principle). Thus F itself is bounded by 1 on the strip.

Note: this is an example of the *Phragmen-Lindelof method*.

2.4.3 Phragmen-Lindelof Method

Generally, if S is unbounded, and f doesn't grow "too fast" in S, then we introduce some h_ε with $h_\varepsilon \to 0$ as $\varepsilon \to 0$ such that $|fh_\varepsilon| < M$ on the boundary of some $S_{\text{bdd}} \subseteq S$, a bounded subregion. Argue that the subregion can be expanded (using that f doesn't grow "too fast") to all of S

Then $|fh_\varepsilon| < M$ on S, so by sending $\varepsilon \to 0$, we get f bounded in S.

Let's consider the specific case of the upper-right quadrant. (See Spring 2015 #8)

Suppose f is bounded on the boundary of the first quadrant by M and $|f(z)| \leq e^{|z|}$ in the interior.

Let $\varepsilon > 0$ and define

$$h_\varepsilon(z) = e^{-\varepsilon\left(e^{-i\pi/4}z\right)^{3/2}}.$$

Note that the 3/2 power and the rotation guarantee that as $z \to \infty$, we have $h_\varepsilon(z) \to 0$ like $e^{|z|^{3/2}}$. Thus $f(z)h_\varepsilon(z)$ is globally bounded in the first quadrant.

Since f is bounded by M on the real and imaginary axes and h_ε is at most 1, we have that fh_ε is bounded by M on some $B(0,R) \cap$ 1st quadrant. Then $|f| \leq M/|h_\varepsilon| \to M$ as $\varepsilon \to 0$. Thus f is bounded globally, so f is constant by Liouville's Theorem.

2.5 Harmonic Functions

2.5.1 (Conjugate) Differentials

Given u harmonic, we write

$$\begin{aligned} du &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \\ {}^*du &= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \\ &= r\frac{\partial u}{\partial r}d\theta \end{aligned}$$

where ${}^*du$ is the *conjugate differential* of u.

If $F = u + iv$, then ${}^*du = dv$ by the Cauchy-Riemann equations. In general, v may not exist.

Note that $\int_\gamma du = 0$ on all cycles homologous to zero, since du is exact (partials commuting). In fact, du is exact, so its integral is zero on any curve (this is just Stokes on the closed curve itself).

Furthermore, $\int_\gamma {}^*du = 0$ on all cycles homologous to zero by harmonicity. This is because ${}^*du$ is closed.

Alternatively, define $f = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ (this is holomorphic!), and then $f\,dz = du + i\,{}^*du$. Then $\int_\gamma f\,dz = 0$ on all cycles homologous to zero, and so we deduce:

$$\int_\gamma {}^*du = 0$$

on all cycles homologous to zero.

2.5.2 An integration result on harmonic functions (like integration by parts)

If u_1 and u_2 are harmonic in a region, then

$$\int_\gamma u_1\,{}^*du_2 - u_2\,{}^*du_1 = 0$$

for all cycles homologous to zero.

Proof: It's enough to prove this on a simply connected domain, on which ${}^*du_i = dv_i$. Then just write

$$u_1 dv_2 - u_2 dv_! = u_1 dv_2 + v_1 du_2 - d(u_2 v_1)$$

and note that the first two terms are $\mathrm{Im}\left((u_1 + iv_1)(du_2 + idv_2)\right)$ which is the imaginary part of a function times an exact form.

2.5.3 When is a harmonic $u = \mathrm{Re}(f)$?

Certainly real parts of holomorphic functions are harmonic (i.e., have Laplacian zero) by the Cauchy-Riemann equations.

Let u be harmonic. We can always construct an analytic function via

$$f = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

(hint: Cauchy-Riemann).

The question is whether or not it has an anti-derivative. In particular, for u harmonic to be the real part of a holomorphic function, we need $f dz = du + i\,{}^*du$ to have an anti-derivative.

Recall that du and ${}^*du$ have zero integral on all curves homologous to zero, so $f dz$ shares this property.

Furthermore, $\int_\gamma du = 0$ on *every* curve. Thus $f dz$ has a well-defined antiderivative F (with $\mathrm{Re}(F) = u$ as desired) if and only if

$$\int_{\gamma_i} {}^*du = 0$$

for each cycle in a homology basis for the domain (i.e., loops around the "holes").

This leads to a theorem: harmonic functions modulo real parts of holomorphic functions (as vector spaces) have dimension equal to the number of cycles in a homology basis.

Proof: if all the periods ($C_i = \int_{\gamma_i} {}^*du$) are zero, we're done. Otherwise, for each γ_i in a homology basis, subtract $C/(2\pi iz)$:

$$\int_{\gamma_i} {}^*du - \frac{C}{2\pi iz}\, dz = 0$$

and so with $f = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ as usual (with $f\, dz = du + i\,{}^*du$), we get

$$\int_{\gamma_i} (f - \frac{C}{2\pi iz}\, dz = 0$$

over every γ_i, and thus $F = \int f - \sum C_i/(2\pi i(z - a_i))$ is a well-defined antiderivative. In particular,

$$u = \operatorname{Re} F + \sum \frac{C_i}{2\pi} \log|z - a_i|$$

for a_i in the "holes" of the homology basis.

2.5.4 Harmonic functions on the punctured disc

Recall a few facts:

1. $\int u_1 {}^*du_2 - u_2 {}^*du_1 = 0$ along curves homologous to zero for u_1, u_2 harmonic.

2. $\int {}^*du = 0$ on curves homologous to zero.

3. ${}^*du = r\frac{\partial u}{\partial r}\, d\theta$ in polar coordinates

These facts combined with $u_1 = u$ and $u_2 = \log r$ give us

$$\log r_1 \int_{C_1} r\frac{\partial u}{\partial r}\, d\theta - \int_{C_1} u\, d\theta = \text{same but with } C_2$$

since the difference of the curves C_1 and C_2 is homologous to zero.

But this means

$$\begin{aligned}\beta &= \int u\, d\theta - \log r \int r\frac{\partial u}{\partial r}\, d\theta \\ &= \int u\, d\theta - \log r \int {}^*du\end{aligned}$$

is constant but so is $\int {}^*du$. (Call this constant α).

Thus $\int u\, d\theta = \alpha \log r + \beta$, the result we desired.

2.5.5 Mean Value Property

Harmonic functions are equal to their averages on balls (or circles).

In fact, a continuous function satisfying $u(z) = ⨍ u(z + re^{i\theta})\, d\theta$ for sufficiently small radii is harmonic.

This implies that if a harmonic function on a punctured disc goes like $\alpha \log r + \beta$ and $\alpha = 0$, then the function is harmonic on the whole disc.

2.5.6 Harnack's Inequality and Principle

1. Harnack's Inequality in a Ball

   If $u \geq 0$ is harmonic on $B(0, R)$, then for all $x \in B(0, r)$ (where $r < R$),

$$\frac{1 - \frac{r}{R}}{1 + \frac{r}{R}} f(0) \leq f(x) \leq \frac{1 + \frac{r}{R}}{1 - \frac{r}{R}} f(0)$$

   or equivalently,

$$\frac{R - r}{R + r} f(0) \leq f(x) \leq \frac{R + r}{R - r} f(0).$$

   Proof: apply Poisson's formula (not the Re version but the other form)

$$f(x) = \frac{1}{2\pi} \int_{|y|=R} \frac{R^2 - r^2}{R|x - y|^2} f(y)\, dy$$

   and bound $R - r \leq |x - y| \leq R + r$. The right hand side is then bounded appropriately (times $⨍ f = f(0)$).

2. General Harnack's Inequality

   Let $u \geq 0$ be harmonic in Ω. Then

$$\sup_K u \leq C \inf_K u$$

   for all u harmonic in Ω and $K \subset\subset \Omega$ where C depends on K and Ω only.

   Proof: Harnack in a ball and a covering argument.

3. Harnack's Principle

   If u_n are harmonic and monotone increasing, then the limit is either infinite everywhere, or finite everywhere.

   Furthermore, the limit is locally uniform and the limiting function is harmonic.

   Proof: If it's infinity anywhere, the Harnack's inequality guarantees it's infinite everywhere (consider the differences which are non-negative!).

   Uniformity is then guaranteed by Harnack's inequality.

   Use Poisson's formula then to obtain harmonicity using uniform convergence.

2.5.7 Representation Formulas for Harmonic Functions

1. Poisson Integral Formula for the Disc

   This is also known as the representation formula for harmonic functions on the disc. If u is harmonic on the disc and continuous up to the boundary (or harmonic on the disc and $R < 1$ and $|a| < R$), we have:

$$\begin{aligned}
f(a) &= \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left(\frac{e^{i\theta}+a}{e^{i\theta}-a}\right) f(e^{i\theta})\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left(\frac{Re^{i\theta}+a}{Re^{i\theta}-a}\right) f(Re^{i\theta})\,d\theta \\
&= \frac{1}{2\pi}\int_{|z|=1} \frac{1-|a|^2}{|z-a|^2} f(z)\,dz \\
&= \frac{1}{2\pi}\int_{|z|=R} \frac{R^2-|a|^2}{|z-a|^2} f(z)\,dz
\end{aligned}$$

   Proof: take $a \in B(0,R)$ onto $0 \in B(0,1)$ via $\frac{R(z-a)}{R^2-\bar{a}z}$. Apply the mean value theorem and struggle with a change of variables.

   Note, the kernel $\operatorname{Re}\left(\frac{e^{i\theta}+a}{e^{i\theta}-a}\right)$ goes to zero as $a \to e^{i\theta'} \neq e^{i\theta}$. Actually, on the circle, at any point other than $e^{i\theta}$, the kernel has value zero. At $e^{i\theta}$, it's undefined. It's in fact equal to $\delta_{e^{i\theta}}$ on the circle in some sense.

   Furthermore, the integral defines a harmonic function whenever f is piecewise continuous.

2. Poisson Integral Formual for the Upper Half Plane

$$f(x+iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+t^2} f(t)\,dt$$

   Use a Möbius transformation of the disc to the upper half plane and transform the Poisson integral for the disc appropriately. This is some messy algebra.

   Alternatively, one can do some trickery with Green's functions.

3. Standard Fact A harmonic function $u : \mathbb{D} \to \mathbb{R}$ is uniformly continuous if and only if it admits a continuous extension to $\partial\mathbb{D}$.

This is a special case of a much better result: let E be such that $\overline{E} = K$ is compact. Then $f : E \to X$ is uniformly continuous if and only if it admits a continuous extension up to K.

If it does admit a continuous extension, it's uniformly continuous since it's living on a compact set.

If it is uniformly continuous, it's easy to define a continuous extension.

2.5.8 Jensen's Formula

Note that $\log|f|$ is harmonic when f is analytic and f has no zeroes. In this case

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(\rho e^{i\theta})|\, d\theta$$

for all ρ where this makes sense. In general, if f has zeroes, we have $\leq$ instead of $=$.

Thus for f with zeroes $\{a_i\}$ in $B(0, r)$, divide by a Blaschke product:

$$g(z) = \prod_{i=1}^{n} \frac{r(z - a_i)}{r^2 - \overline{a_i} z}$$

noting that $|g| = 1$ on $|z| = r$ and that f and g share zeroes inside the disc. Apply the mean value formula to get

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})| &= \log|f(0)| - \log|g(0)| \\ &= \log|f(0)| - \sum_{i=1}^{n} \log\frac{r}{|a_i|} \end{aligned}$$

2.5.9 Poisson-Jensen Formula

Just apply the Poisson formula to $\log|f/g|$ above to calculate the value at a point other than 0.

$$\log|f(z)| = -\sum_{i=1}^{n} \log\left|\frac{r^2 - \overline{a_i} z}{r(z - a_i)}\right| + \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(\frac{re^{i\theta} + z}{re^{i\theta} - z}\right) \log|f(re^{i\theta})|\, d\theta$$

2.5.10 Green's Functions

Suppose Ω has finite connectivity and $z_0 \in \Omega$. Solving the Dirichlet problem in Ω with boundary values $\log|\zeta - z_0|$ yields $G(z)$.

Then $g(z) = g(z, z_0) = G(z) - \log|z - z_0|$ is the *Green's function* of Ω with pole at z_0. This is harmonic except at z_0. In a neighborhood of z_0, $g(z)$ differs from a harmonic function by $-\log|z - z_0|$. These properties uniquely determine $g(z)$.

1. Properties

   (a) Note that $g(z, w) = g(w, z)$. Consider $g_1 = g(x, z)$ and $g_2 = g(x, w)$. Using $\int g_1{}^* dg_2 - g_2{}^* dg_1 = 0$ on curves homologous to zero and considering $G_1 = g_1 + \log|x - z|$, some calculation can be done to show that $g_1 = g_2$.

   (b) In the sense of distributions $\triangle g = \delta$, the Dirac delta, and to solve the Dirichlet problem with boundary values f, consider $g * f$, convolved on the boundary. Then, formally, $\triangle(g * f)(z) = \delta * f = f(z)$.

2.5.11 Riemann Mapping Theorem

Every simply connected domain that is not $\mathbb{C}$ is conformally equivalent to a disc. It is unique up to the normalization that $f(z_0) = 0$ and $\mathbb{R} \ni f'(z_0) > 0$ for some point z_0.

Proof: Let U be the domain but with smooth boundary. Let $u(z) = -\log|z - z_0|$ on the boundary and solve the Dirichlet problem. We get $g(z) = u(z) + iv(z)$. Write $f(z) = (z - z_0)\exp(g(z))$. Then $|f| = 1$ on the boundary. Furthermore, there's only one zero, so by simple connectedness, the function is injective. (This may take a little clarification).

Alternatively, Ahlfors describes a difficult method involving a family of univalent functions into the disc and maximimzing $f'(z_0)$ among this family. It's difficult to show this family is non-empty and that the maximum is as desired.

Either way, the proof is very unlikely to be asked, but the result is absolutely critical to know. It's a convenient way to turn a mess multiply-connected domain into one bounded by Jordan curves, for instance.

2.5.12 Caratheodory's Theorem

A conformal map from the unit disc to a region bounded by a Jordan curve extends continuously to the boundary and is a homeomorphism between boundaries.

The proof is not necessary (and is long), but the result is extremely handy in a pinch.

2.5.13 Schwarz Reflection Principle

If v is continuous on the upper half-plane up to the boundary, harmonic on the upper half plane, and zero on the boundary, then

$$v(\bar{z}) = -v(z)$$

is a harmonic extension.

If v is the imaginary part of an analytic function, you can extend via $f(z) = \overline{f(\overline{z})}$.

After some conformal mappings, Schwarz reflection can be done about other curves. One application is to Schwarz reflection about a circle, used in classifying conformal maps between annuli.

2.6 Subharmonic Functions

2.6.1 Definition

A real continuous function v defined in Ω is subharmonic if for every harmonic u defined in $\omega \subseteq \Omega$, $v - u$ satisfies the maximum principle in ω.

That is to say, $v - u$ doesn't attain a maximum in the interior.

This works for upper semi-continuous functions too, functions with values above the limit at each point, i.e., $\limsup_{a \to b} f(a) \leq f(b)$. Alternatively, $\{f < \alpha\}$ is open for each α.

Equivalently, a continuous function v is subharmonic iff

$$v(z_0) \leq ⨍_{\partial B(z_0,r)} v \, d\theta$$

for sufficiently small r.

2.6.2 Peron's Method

Consider the space of functions that are subharmonic whose limsup is at most $f(\zeta)$ on the boundary of a domain (where f is continuous). If Ω has a "nice boundary" (a *barrier* at each point), then the sup of such functions (which is automatically subharmonic) is in fact harmonic.

Most reasonable domains have a barrier at each point.

2.7 Residue Calculus

2.7.1 Basic Types of Residue Calculations

What follows is a list of examples of integrals that can be evaluated with residues.

1. All integrals of the form

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$$

   where R is rational and easy to evaluate.

   Solution: substitute $z = e^{i\theta}$ and get

$$-i \int_{|z|=1} R\left(\frac{1}{2}(z + \frac{1}{z}), \frac{1}{2i}(z - \frac{1}{z})\right) \frac{1}{z}\, dz$$

   since $\cos\theta = \operatorname{Re} z = \frac{1}{2}(z + \bar{z}) = \frac{1}{2}(z + 1/z)$ on the unit circle. Furthermore, $d\theta = (-i/z)dz$.

   Example:

$$\begin{aligned} \int_0^{\pi} \frac{1}{a + \cos\theta}\, d\theta &= \frac{1}{2}\int_0^{2\pi} \frac{1}{a + \cos\theta} \\ &= \frac{1}{2i}\int_{|z|=1} \frac{2}{2a + z + 1/z}\frac{1}{z}\, dz \\ &= \frac{1}{i}\int_{|z|=1} \frac{1}{z^2 + 2az + 1}\, dz \end{aligned}$$

   so we do partial fractions and use the residue theorem.

$$\frac{1}{z^2 + 2az + 1} = \frac{A}{z - a - \sqrt{a^2 - 1}} + \frac{B}{z - a + \sqrt{a^2 - a}}$$

   Since only one of these roots is in the unit disc, we push through some algebra and get $\pi/\sqrt{a^2 - 1}$.

2. An integral of the form

$$\int_{-\infty}^{\infty} R(x)\, dx$$

   where R is rational converges iff R has denominator degree at least two more than numerator and no pole on $\mathbb{R}$.

This degree condition guarantees that the integral on a large semicircle vanishes. We end up with (considering a line $-r$ to r then a semicircle back and taking a limit)

$$\int_{-\infty}^{\infty} R(x)\,dx = 2\pi i \sum_{y>0} \operatorname{Res} R(z)$$

Example:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{x^2+1}{(x^2+4)^2}\,dx &= 2\pi i \operatorname{Res}(2i)\\ &= 2\pi i \frac{-5i}{32} = 5\pi/32\end{aligned}$$

3. An integral of the form

$$\int_{-\infty}^{\infty} R(x)e^{ix}\,dx$$

where R is rational and has a zero of order 2 at infinity.

Note that

$$\left|e^{iz}\right| = e^{-y}$$

is bounded in the upper half-plane, so the integral over that large semicircle still goes to zero and we still get

$$\int_{-\infty}^{\infty} R(x)e^{ix}\,dx = 2\pi i \sum_{y>0} \operatorname{Res}(R(z)e^{iz})$$

If R has a simple order at infinity, we can use a rectangular path with left endpoint a, right endpoint b, and height h. The right side is bounded by

$$\begin{aligned}\int_0^h e^{-y}|R(z)|\,dy &\lesssim \int_0^h e^{-y}\frac{1}{|z|}\,dy\\ &\le \frac{1}{b}\int_0^y e^{-y} \lesssim 1/b\end{aligned}$$

Similarly, this holds on the left side, and the upper horizontal integral is bounded by $e^{-h}(a+b)/h$. In the right order, send $h \to \infty$ then $a, b \to \infty$ and we get the same formula as before.

Example:

$$\int_{-\infty}^{\infty} \frac{x}{x^2+1} \sin(x)\, dx = \operatorname{Im} \int_{-\infty}^{\infty} \frac{x}{x^2+1} e^{ix}\, dx$$
$$= \operatorname{Im}\left(2\pi i \operatorname{Res}_i \left(\frac{ze^{iz}}{z^2+1}\right)\right)$$
$$= \operatorname{Im}\left(2\pi i \frac{ie^{-1}}{2i}\right) = \pi/e$$

4. An integral of the form

$$\int_0^\infty x^\alpha R(x)\, dx$$

where $0 < \alpha < 1$, R has at most a simple pole at 0, and at least a zero of order 2 at ∞.

We have two options here.

(a) Substitute $x = t^2$ and instead get

$$2\int_0^\infty t^{2\alpha+1} R(t^2)\, dt$$

after the change of variables. Now the integrand is (almost) odd.

Let's calculate

$$\int_{-\infty}^{\infty} z^{2\alpha+1} R(z^2)\, dz$$

by integrating on a large and small semi-circle (both integrals go to zero) and then $-R \to -r$ and $r \to R$.

We end up with

$$\int_{-\infty}^{\infty} z^{2\alpha+1} R(z^2)\, dz = (1 - e^{2\pi i\alpha}) \int_0^\infty z^{2\alpha+1} R(z^2)\, dz$$

where the first integral is $2\pi i \sum_{\mathrm{Im}>0} \operatorname{Res}(z^{2\alpha+1} R(z^2))$.

Note that the residues of $z^{2\alpha+1} R(z^2)$ in the upper half-plane are the same as those of $z^\alpha R(z)$ in the whole plane.

(b) We can ignore the initial subtitution by consider a keyhole integral.

We pick the branch of z^α whose argument lies between 0 and $2\pi\alpha$ and cut along the positive real axis.

Our keyhold integral goes $r + i\varepsilon$ around CCW to $r - i\varepsilon$, then out to $R - i\varepsilon$, CW to $R + i\varepsilon$, then back to $r + i\varepsilon$.

The two straight line integrals converge to the desired $\int_0^\infty$, but differ by an $e^{2\pi i\alpha}$ and a sign. We get the same answer as before.

Remark: The integral $\int_0^\pi \log\sin\theta\, d\theta$ appears in Ahlfors, but seems like it won't appear on the exam as it's a messy solution.

2.8 Infinite Products and Sums

2.8.1 Convergence

The infinite product $\prod_1^\infty(1 + a_n)$ converges (assuming $1 + a_n = 0$ only finitely many times) iff $\sum \log(1 + a_n) < \infty$, where log is the principal branch.

The product $\prod(1 + a_n)$ converges *absolutely* iff $\sum|a_n|$ converges, because $\frac{\log(1+z)}{z} \to 1$ as $z \to 0$ and thus

$$(1 - \varepsilon)|a_n| < \left|\log(1 + a_n)\right| < (1 + \varepsilon)|a_n|$$

for large n can be made to hold.

2.8.2 Weierstrass Products

Every function with arbitrarily prescribed zeroes (going to ∞) can be written in the form

$$f(z) = z^m e^{g(z)} \prod_{n=1}^\infty \left[\left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}}\right]$$

Corollary: every meromorphic function in the whole plane is the quotient of two entire functions (just match zeroes).

2.8.3 Genus

The *genus* of a product is the smallest h such that if $m_n = h$ for each n, the product still converges. Then

$$\prod_{n=1}^\infty \left[\left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \cdots + \frac{1}{h}\left(\frac{z}{a_n}\right)^h}\right]$$

is called the *canonical product*.

The genus h is also equal to the smallest non-negative integer for which

$$\sum \left(\frac{1}{|a_n|}\right)^{h+1}$$

converges.

The *genus* of a function is the maximum of the genus of the product and the degree of $g(z)$ as a polynomial in the Weierstrass product. (If g isn't a polynomial, the genus is ∞).

2.8.4 Examples of Genus

1. A function of the form

   $$Cz^m \prod \left(1 - \frac{z}{a_n}\right)$$

   has genus 0 if $\sum 1/|a_n| < \infty$ (so that the product is well-defined).

2. A function of the form

   $$Cz^m e^{\alpha z} \prod (1 - z/a_n)\, e^{z/a_n}$$

   with $\sum 1/|a_n|^2 < \infty$ but $\sum 1/|a_n| = \infty$ is genus 1.

   This is because the genus of the product is 1 and the polynomial g has degree 1.

3. A function of the form

   $$Cz^m e^{\alpha z} \prod (1 - z/a_n)$$

   with $\sum 1/|a_n| < \infty$ still has genus 1, because $\deg g = 1$.

2.8.5 Order

Define

$$M(r) = \max_{|z|=r} |f(z)|.$$

Then the *order* of f is

$$\lambda = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r},$$

i.e., the smallest λ such that $M(r) \lesssim e^{r^{\lambda+\varepsilon}}$ for any $\varepsilon > 0$ as soon as r is sufficiently large.

2.8.6 Order and Genus

We have $h \leq \lambda \leq h + 1$. The proof is a pain and seems incredibly unlikely to appear.

2.8.7 Mittag-Leffler Theorem / Partial Fractions

If f is meromorphic with poles p_ν, the singular part looks like $P_\nu\left(\frac{1}{z-b_\nu}\right)$ where ν is some polynomial.

If we pick the right analytic functions p_ν we can make $\sum(P_\nu - p_\nu)$ converge.

Theorem: Let $b_\nu \to \infty$ and $P_\nu(\zeta)$ be polynomials without constant term. Then there are functions meromorphic in the plane with singular parts

$$P_\nu\left(\frac{1}{z-b_\nu}\right)$$

at the poles b_ν and all such functions can generally be written as

$$f = \sum\left(P_\nu\left(\frac{1}{z-b_\nu}\right) - p_\nu(z)\right) + g(z)$$

where p_ν are polynomials, g is analytic and entire.

Proof: take power series expansion for $P_\nu(1/(z-b_\nu))$ about the origin and choose $p_\nu(z)$ a partial sum of this series, ending in the term of degree n_ν. The difference can be estimated by Taylor's theorem:

$$\left|P_\nu\left(\frac{1}{z-b_\nu}\right) - p_\nu(z)\right| \leq 2M\left(\frac{2|z|}{|b_\nu|}\right)^{n_\nu+1}$$

in $|z| \leq |b_\nu|/4$ if P_ν is bounded by M in twice that disc.

Thus by choosing n_ν sufficiently large, we can guarantee absolute convergence in the whole plane except at the poles.

We also get uniform convergence in the disc.

This is commonly applied to the series expansion for $\sin^2 \pi z$.

2.9 Useful Results to Remember

2.9.1 Classify Conformal Maps Between Annuli

Let $f : A_R \to A_S$ be conformal where $A_R = B(0,R) \setminus \overline{B(0,1)}$. Then $R = S$.

Proof:

Without loss of generality, f sends the unit circle to itself. Since f is non-vanishing analytic, $\log|f|$ is harmonic and takes boundary values 0 and $\log S$.

Note that $\frac{\log(S)}{\log(R)} \log|z|$ is also harmonic and has the same boundary values. Since the Dirichlet problem can be solved on the annulus, $\log|f|$ is determined uniquely by its boundary values, so

$$\log|f(z)| = \frac{\log S}{\log R} \log|z|$$

and so $|f(z)| = |z|^{\log(S)/\log(R)} = \left|z^{\log S/\log R}\right|$.

Since f and z^α are both analytic (taking a branch cut) and have the same absolute value, they're equal up to rotation.

Thus $f(z) = Cz^\alpha$. But f is injective, so $\alpha = 1$ and $S = R$, so f is in fact a rotation.

2.9.2 Examples of Conformal Maps

1. $z \mapsto 1/z$ sends $\{\mathrm{Re} > 1/2\} \to B(1,1)$.
2. $\frac{z-1}{z+1}$ sends $\{\mathrm{Re} > 0\} \to B(0,1)$ and sends $B(0,1) \to \{\mathrm{Re} < 0\}$.
3. $z + 1/z$ sends $B(0,1)$ or its complement to $\mathbb{C} \setminus [-2,2]$.
4. exp sedns the vertical strip to $\mathbb{C} \setminus \{0\}$.
5. $\frac{z-a}{1-\bar{a}z}$ is the standard form of an automorphism of the unit disc.
6. Log (principal branch, i.e., negative reals omitted) sends $\mathbb{C} \setminus \{x < 0\} \to [0, 2\pi] \oplus i\mathbb{R}$.
7. x^n conformally maps a wedge onto the half-plane

2.9.3 Proof of the Poisson-Jensen Formula

See notes above.

2.9.4 A basis for the space of real parts of holomorphic mod harmonic functions

See notes above.

2.9.5 Examples of Sums and Products

1. Let's consider $\frac{\pi^2}{\sin^2 \pi z}$ with Mittag-Lefler. The singular parts are $1/(z-n)^2$. We write

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{-\infty}^{\infty} \frac{1}{(z-n)^2} + g(z)$$

where we claim $g(z)$ is zero. Note that everything is periodic with period 1.

Note that both $\pi^2/\sin^2 \pi z$ and the sum go to 0 in the strip as $y \to \infty$. Thus $g(z)$ is periodic and bounded, hence constant. Furthermore it must be zero since $0 + 0 = 0$.

2. Integrating the above in z (cancelling minus signs) we get

$$\begin{aligned}\pi \cot(\pi z) &= \frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right) \\ &= \lim_{m \to \infty} \sum_{n=-m}^{m} \frac{1}{z-n} \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}\end{aligned}$$

whose terms converge by comparison test and whose derivatives match the previous equality. Note that the LHS and RHS are both odd so the integration constant $+C$ must be zero.

3. Now we consider

$$\lim_{m \to \infty} \sum_{n=-m}^{m} \frac{(-1)^n}{z-n}.$$

Separating odd and even terms and comparing to what we know above for cot and applying magic trig identities, we get

$$\begin{aligned}\frac{\pi}{\sin \pi z} &= \lim_{m \to \infty} \sum_{n=-m}^{m} \frac{(-1)^n}{z-n} \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \frac{2z}{z^2 - n^2}\end{aligned}$$

4. We have

$$\sin \pi z = \pi z \prod_{n \neq 0} \left(1 - \frac{z}{a_n}\right) e^{z/n}$$

by considering the zeroes and then taking the logarithmic derivative to see what the $\exp(g(z))$ term is (constant). This is a formula to memorize.

5. With more difficulty, we have

$$\begin{aligned} \cos \pi z &= \prod_{n \in 2\mathbb{Z}+1} \left(1 - \frac{2z}{n}\right) e^{2z/n} \\ &= \prod_{j=0}^{\infty} \left(1 - \left(\frac{z}{j+1/2}\right)^2\right), \end{aligned}$$

by using $\cos z = \sin(2z)/2\sin(z)$ and the previous product.

2.9.6 Poisson formula for harmonic functions in the disc

Show that a harmonic function $u : \mathbb{D} \to \mathbb{R}$ is uniformly continuous if and only if it admits the representation

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re}\left(\frac{e^{i\theta} + z}{e^{i\theta} - z}\right) f(e^{i\theta})\, d\theta$$

with $f : \partial\mathbb{D} \to \mathbb{R}$ continuous.

Alternatively, we have the formula

$$u(z) = \frac{1}{2\pi} \int_{|w|=R} \frac{R^2 - r^2}{R|w-z|^2} u(w)\, dw$$

This lets us solve the Dirichlet problem on the disc.

1. Proof

   It is a standard fact that u is uniformly continuous if and only if it admits a continuous extension to $\partial\mathbb{D}$.

   If u has a continuous extension to $\partial\mathbb{D}$, just apply the Poisson integral formula.

   (To prove it, apply the mean value formula after a conformal map $w \mapsto \frac{w+z}{1+\bar{z}w}$ and simplify the change of variables.)

Conversely, suppose u has the above representation. Need to show that u extends to f continuously.

Fix $e^{i\theta_0} \in \partial\mathbb{D}$. Fix $\varepsilon > 0$. Pick δ_1 such that $|\theta - \theta_0| < \delta$ implies $\left|f(e^{i\theta_0}) - f(e^{i\theta})\right| < \varepsilon$ by continuity of f.

Since $\partial\mathbb{D}$ is compact, let $M = \max\left|f(e^{i\theta})\right|$. Now finally pick $\delta > 0$ small enough that

$$\left|z - e^{i\theta}\right| < \delta$$

and $|\theta - \theta_0| \geq \delta_1$ implies that Re(stuff) $< \varepsilon/2M$.

Note that $\int_0^{2\pi}$ Re(stuff) $= 2\pi$, and so do a classic "add in an integral" estimate

$$\left|u(z) - f(e^{i\theta_0})\right| = \frac{1}{2\pi}\left|\int \mathrm{Re(stuff)}\left(f(e^{i\theta}) - f(e^{i\theta_0})\right)\, d\theta\right|$$

Near θ_0, use that f gets small. Away from θ_0, use that the real part of stuff gets small.

2.9.7 (L^p) Integrability of C

2.9.8 S17 Problem 9a and maybe 9b

2.9.9 Entire function bounded by a polynomial is polynomial

Suppose $|f(z)| \leq C|z|^n$ for sufficiently large $|z|$.

If we have this inequality everywhere, Liouville's theorem guarantees $f = \alpha z^n$ by looking at the quotient f/Cz^n.

In this case, we apply Cauchy's integral formula:

$$\begin{aligned}\left|f^{(k)}(0)\right| &= \frac{k!}{2\pi i}\left|\int_{C_R} \frac{f(z)}{z^{k+1}}\, dz\right| \\ &\lesssim \frac{R^n}{R^k}\end{aligned}$$

which goes to zero if $n > k$, so f has $(k+1)$-st and later derivatives equal to zero. This argument can be shifted around to apply to any point in the plane.